Proofs Involving Sets #5 (Proving the Contrapositive) - Solutions

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Instructions. Prove the following by proving the contrapositive.

1. $\underbrace{A \subseteq B}_{p} \Rightarrow \underbrace{(A \cap B) = A}_{q}$

   Proof. We will prove the contrapositive, $\underbrace{(A \cap B) \neq A}_{\sim q} \Rightarrow \underbrace{A \nsubseteq B}_{\sim p}$.

   Let the hypothesis be given. (i.e., Suppose that $(A \cap B) \neq A$).

   $\Rightarrow$ either $(A \cap B) \nsubseteq A$ or $A \nsubseteq (A \cap B)$. (Otherwise, if each set were a subset of the other, the sets would be equal, contrary to our hypothesis.)

   Since $(A \cap B) \subseteq A$ (always!) this leaves, as the only possibility, $A \nsubseteq (A \cap B)$.

   $\Rightarrow \exists x \in A$ such that $x \notin (A \cap B)$

   $\Rightarrow x \in A$ and either $x \notin A$ or $x \notin B$.

   i.e., $\underbrace{x \in A \text{ and } x \notin A}_{\text{impossible}}$, or $x \in A$ and $x \notin B$

   $\Rightarrow x \in A$ and $x \notin B$.

   i.e., A has an element that B doesn't.

   Hence, $A \nsubseteq B$.

   We have shown that $(A \cap B) \neq A \Rightarrow A \nsubseteq B$. ■

2. $\underbrace{A \subseteq B}_{p} \Rightarrow \underbrace{(A \cup B) = B}_{q}$

Proof. We will prove the contrapositive, $\underbrace{(A \cup B) \neq B}_{\sim q} \Rightarrow \underbrace{A \nsubseteq B}_{\sim p}$.

Let the hypothesis be given. (i.e., Suppose that $(A \cup B) \neq B$.

$\Rightarrow$ either $(A \cup B) \nsubseteq B$ or $B \nsubseteq (A \cup B)$. (Otherwise, if each set were a subset of the other, the sets would be equal, contrary to our hypothesis.)

Since $B \subseteq (A \cup B)$ (always!) this leaves, as the only possibility, $(A \cup B) \nsubseteq B$.

$\Rightarrow \exists x \in (A \cup B)$ such that $x \notin B$

$\Rightarrow \exists x \ni (x \in A$ or $x \in B)$ and $x \notin B$.

i.e., $x \in A$ and $x \notin B$, or $\underbrace{x \in B \text{ and } x \notin B}_{\text{impossible}}$

$\Rightarrow x \in A$ and $x \notin B$.

i.e., A has an element that B doesn't.

Hence, $A \nsubseteq B$.

We have shown that $(A \cup B) \neq B \Rightarrow A \nsubseteq B$. ■

3. $\underbrace{(A \cap B) = \emptyset}_{p} \Rightarrow \underbrace{A \subseteq B^c}_{q}$

Proof. We will prove this by proving the contrapositive, $\underbrace{A \nsubseteq B^c}_{\sim q} \Rightarrow \underbrace{(A \cap B) \neq \emptyset}_{\sim p}$.

Let the hypothesis be given. (i.e., suppose that $A \nsubseteq B^c$).

$\Rightarrow \exists x \in A$ such that $x \notin B^c$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A \cap B$

$\Rightarrow (A \cap B) \neq \emptyset$

We have shown that $A \nsubseteq B^c \Rightarrow (A \cap B) \neq \emptyset$. ■