MTH 1126 - Test #1 - Solutions

Spring 2008

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Instructions. Show CLEARLY how you arrive at your answers.

1. $\int \left(4x^3+3x^2-6x+3\sqrt{x}+\sec^2(x)\right)dx \underbrace{=}_{\nwarrow \text{ re-write } \nearrow} \int \left(4x^3+3x^2-6x+3x^{\frac{1}{2}}+\sec^2(x)\right)dx$

$= 4\left[\frac{x^4}{4}\right]+3\left[\frac{x^3}{3}\right]-6\left[\frac{x^2}{2}\right]+3\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]+\tan(x)+C = x^4+x^3-3x^2+2x^{\frac{3}{2}}+\tan(x)+C$

i.e., $\int \left(4x^3+3x^2-6x+3\sqrt{x}+\sec^2(x)\right)dx = x^4+x^3-3x^2+2x^{\frac{3}{2}}+\tan(x)+C$

2. Use the "$f-g$" method to compute the area bounded by the graphs of $f(x)=x^3$ and $g(x)=4x$.

First, graph the functions and find the points of intersection.

To find the points of intersection, set $f(x)=g(x)$.

$f(x)=x^3=4x=g(x)$

$\Rightarrow x^3=4x$

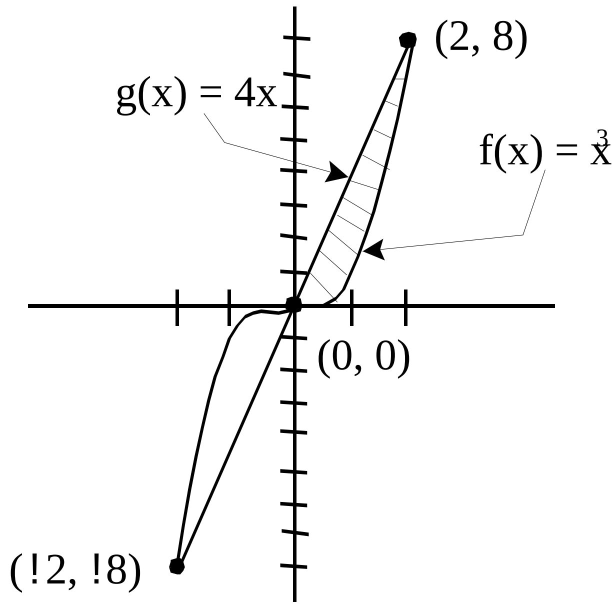
$\Rightarrow x^3-4x=0$

$\Rightarrow x\left(x^2-4\right)$

$\Rightarrow x(x+2)(x-2)=0$

$\Rightarrow x=0,\ x=-2;\ x=2$

The points of intersection are: $(-2,-8)$, $(0,0)$, and $(2,8)$.



Since $4x \geq x^3$ over the interval $[0, 2]$, the area of the bounded region is given by

$\int_0^2 (4x - x^3)\, dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = \left(2\,(2)^2 - \frac{1}{4}\,(2)^4\right) - \left(2\,(0)^2 - \frac{1}{4}\,(0)^4\right) = 4$

Area $= 4$

3. Suppose that $\int_2^8 (f(x) + g(x))\, dx = 10$; $\int_2^8 g(x)\, dx = 5$; and that $\int_4^2 f(x)\, dx = 4$. Compute $\int_4^8 f(x)\, dx$.

First, we need to find the value of $\int_2^8 f(x)\, dx$.

Since $\int_2^8 (f(x) + g(x))\, dx = \int_2^8 f(x)\, dx + \int_2^8 g(x)\, dx$, it follows that

$\int_2^8 f(x)\, dx = \int_2^8 (f(x) + g(x))\, dx - \int_2^8 g(x)\, dx = 10 - 5 = 5$

i.e., $\int_2^8 f(x)\, dx = 5$

Next observe that $\int_4^2 f(x)\, dx + \int_2^8 f(x)\, dx = \underbrace{\int_4^8 f(x)\, dx}_{\text{this is what we want}}$

Thus, $\int_4^8 f(x)\, dx = \int_4^2 f(x)\, dx + \int_2^8 f(x)\, dx = 4 + 5 = 9$

i.e., $\int_4^8 f(x)\, dx = 9$

4. Compute: $\int_{x=0}^{x=1} (2x^3+3)^3 x^2 \, dx =$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes. $(2x^3+3)^3$

      Let $u = 2x^3 + 3$

      b. Is there an "approximate function/derivative pair"?

      Yes. $(2x^3+3) \to x^2$

      Let $u = 2x^3 + 3$

   2. Compute du

   $$\boxed{\begin{array}{rcl} u & = & 2x^3+3 \\ \Rightarrow \quad \frac{du}{dx} & = & 6x^2 \\ \Rightarrow \quad du & = & 6x^2 dx \\ \Rightarrow \quad \frac{1}{6}du & = & x^2 dx \end{array}}$$

   $$\boxed{\begin{array}{l} \text{When } x=0,\ u = 2x^3+3 = 3 \\ \text{When } x=1,\ u = 2x^3+3 = 5 \end{array}}$$

   3. Analyze in terms of u and du.

   $$\int_{x=0}^{x=1} \underbrace{\left(2x^3+3\right)^3}_{u^3} \underbrace{x^2\,dx}_{\frac{1}{6}du} = \int_{u=3}^{u=5} u^3 \tfrac{1}{6} du = \tfrac{1}{6} \int_{u=3}^{u=5} u^3 du$$

   4. Integrate in terms of u

   $$\tfrac{1}{6} \int_{u=3}^{u=5} u^3 du = \tfrac{1}{6} \left[\tfrac{u^4}{4} \right]_{u=3}^{u=5} = \tfrac{1}{24}\left(5^4 - 3^4\right) = \tfrac{68}{3}$$

   i.e., $\int_{x=0}^{x=1} (2x^3+3)^3 x^2 \, dx = \frac{68}{3}$

5. Find the area bounded by the graphs of $f(x) = 1 - x^2$ and $g(x) = x^2 - 1$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

   1. First, graph the functions and find the points of intersection.

   To find the points of intersection, set $f(x) = g(x)$.

   $f(x) = 1 - x^2 = x^2 - 1 = g(x)$

   $\Rightarrow 1 - x^2 = x^2 - 1$

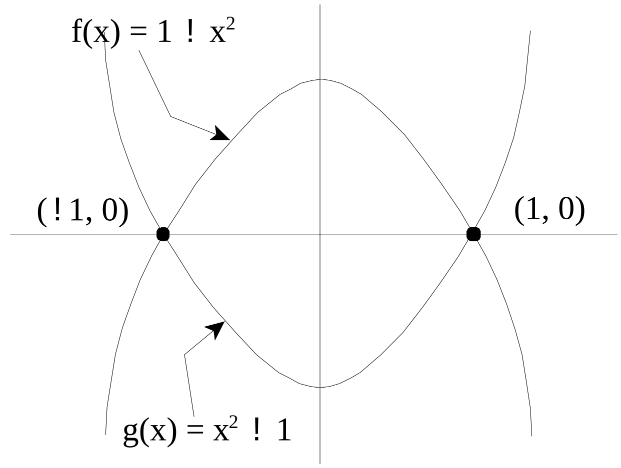
   $\Rightarrow 2 - 2x^2 = 0$

   $\Rightarrow 1 - x^2 = 0$

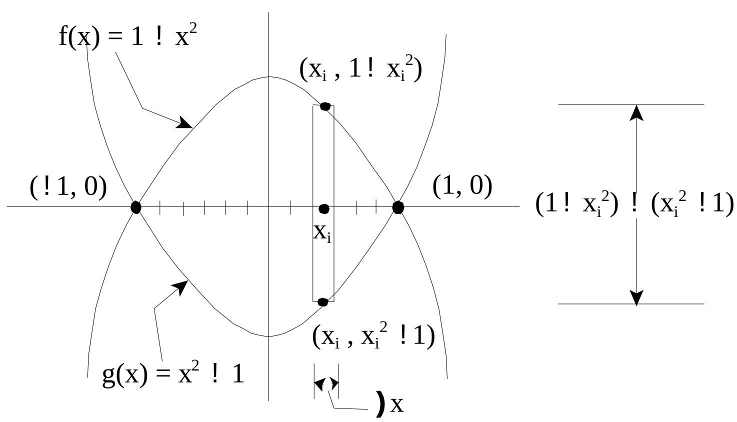
   $\Rightarrow (x+1)(x-1) = 0$

   $\Rightarrow x = -1;\ x = 1$

Points of intersection are $(-1, 0)$ and $(1, 0)$.



2. Partition the interval spanned by the region into sub-intervals of length Δx.
3. Above the i^{th} subinterval, inscribe a rectangle of width Δx.



Height of the i^{th} rectangle $= [(1 - x_i^2) - (x_i^2 - 1)] = 2 - 2x_i^2$

Width of the i^{th} rectangle $= \Delta x$

Area of the i^{th} rectangle $= (2 - 2x_i^2)\,\Delta x$

4. Approximate the area of the region by adding up the areas of the rectangles.

   Area $\approx \sum_{i=1}^{n} (2 - 2x_i^2)\,\Delta x$

5. Let $\Delta x \to 0$.

   Area $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} (2 - 2x_i^2)\,\Delta x = \int_{-1}^{3} (2 - 2x^2)\,dx = \left[2x - \frac{2}{3}x^3\right]_{-1}^{1} =$

   $\left(2\,(1) - \frac{2}{3}\,(1)^3\right) - \left(2\,(-1) - \frac{2}{3}\,(-1)^3\right) = \frac{8}{3}$

   Bounded area $= \frac{8}{3}$

6. Compute: $\int \tan(3x)\sec^2(3x)\ dx =$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes. $\tan(3x)$ also $\sec(3x)$ also $\sec^2(3x)$

      Let $u = 3x$ or $u = \sec(3x)$

      b. Is there an "approximate function/derivative pair"?

      Yes. $\underbrace{\tan(3x)}_{\text{function}} \rightarrow \underbrace{\sec^2(3x)}_{\text{deriv}}$

      Let $u = \tan(3x)$

   Remark 1 *Since criteria a and b yield different choices for u, we will go with criterin b, the "approximate function/derivative pair," and we will let* $u = \tan(3x)$

   2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & \tan(3x) \\ \Rightarrow & \frac{du}{dx} & = & 3\sec^2(3x) \\ \Rightarrow & du & = & 3\sec^2(3x)\,dx \\ \Rightarrow & \frac{1}{3}du & = & \sec^2(3x)\,dx \end{array}}$$

   3. Analyze in terms of u and du.

$$\int \underbrace{\tan(3x)}_{u}\underbrace{\sec^2(3x)\ dx}_{\frac{1}{3}du} == \int u\tfrac{1}{3}du = \tfrac{1}{3}\int u\,du$$

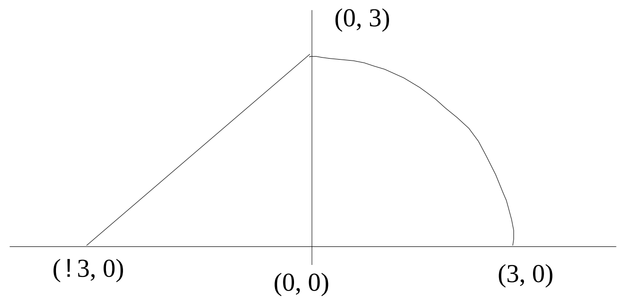
   4. Integrate in terms of u

$$\tfrac{1}{3}\int u\,du = \tfrac{1}{3}\tfrac{u^2}{2} + C = \tfrac{1}{6}u^2 + C$$

   5. Re-write in terms of x

$$\int \tan(3x)\sec^2(3x)\ dx = \underbrace{\frac{1}{6}\tan^2(3x) + C}_{\frac{1}{6}u^2 + C}$$

7. The graph of $f(x)$ is shown below. Compute $\int_{-3}^{3} f(x)\, dx$



Observe that between $x = -3$ and $x = 3$, $f(x) \geq 0$. Therefore, $\int_{-3}^{3} f(x)\, dx$ is equal to the area bounded by the graph of $f(x)$ and the x-axis, between $x = -3$ and $x = 3$.

Thus, $\int_{-3}^{3} f(x)\, dx = (\text{area of triangle with base and height} = 3) + \left(\text{area of } \frac{1}{4} \text{ circle of radius } 3\right)$

$= \frac{1}{2} b \cdot h + \frac{1}{4}\pi r^2 = \frac{1}{2}(3)(3) + \frac{1}{4}\pi(3)^2 = \frac{9}{2} + \frac{9\pi}{4} = \frac{18+9\pi}{4}$