# Integrals and Natural Logarithms #5 - Solutions

Fall 2013

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1. Compute: $\int\left(2x^5-6x^3+4x+8\right)dx = \int\left(2x^5-6x^3+4x+8\right)dx = 2\left[\frac{x^6}{6}\right]-6\left[\frac{x^4}{4}\right]+4\left[\frac{x^2}{2}\right]+8\left[x\right]+C$

   $= \frac{1}{3}x^6 - \frac{3}{2}x^4 + 2x^2 + 8x + C$

   i.e., $\int\left(2x^5-6x^3+4x+8\right)dx = \frac{1}{3}x^6 - \frac{3}{2}x^4 + 2x^2 + 8x + C$
   Don't forget the "+C"

2. Compute: $\int\left(6\sec^2(x) - 3\csc^2(x)\right)dx =$

   $\int\left(6\sec^2(x) - 3\csc^2(x)\right)dx = 6\left[\tan(x)\right] - 3\left[-\cot(x)\right] + C$

   i.e., $\int\left(6\sec^2(x) - 3\csc^2(x)\right)dx = 6\tan(x) + 3\cot(x) + C$
   Don't forget the "+C"

3. Compute: $\int_{x=-2}^{x=0}\left(3x^2+3x+3\right)dx =$

   $$\int_{x=-2}^{x=0}\underbrace{\left(3x^2+3x+3\right)}_{f(x)}dx = \underbrace{\left[3\left(\frac{x^3}{3}\right)+3\left(\frac{x^2}{2}\right)+3x\right]_{x=-2}^{x=0}}_{F(x)} = \underbrace{\left[x^3+\frac{3}{2}x^2+3x\right]_{x=-2}^{x=0}}_{F(x)}$$

   $$= \underbrace{\left[(0)^3+\frac{3}{2}(0)^2+3(0)\right]}_{F(0)} - \underbrace{\left[(-2)^3+\frac{3}{2}(-2)^2+3(-2)\right]}_{F(-2)} = 8$$

   i.e., $\int_{x=-2}^{x=0}\left(3x^2+3x+3\right)dx = 8$

4. Compute: $\int \frac{(4x^3+3)}{\sqrt{3x^4+9x}}dx \quad = \quad \int \frac{(4x^3+3)}{(3x^4+9x)^{\frac{1}{2}}}dx \quad = \quad \int \left(3x^4+9x\right)^{-\frac{1}{2}}\left(4x^3+3\right)\,dx$

↖ re-write ↗ ↖ re-write ↗

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(3x^4+9x\right)^{-\frac{1}{2}}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3x^4+9x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3x^4+9x\right)}_{\text{function}} - - - - - \rightarrow \underbrace{\left(4x^3+3\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3x^4+9x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of $u$ ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute $du$

$$\begin{aligned} u &= 3x^4+9x \\ \Rightarrow \frac{du}{dx} &= 12x^3+9 \\ \Rightarrow du &= \left(12x^3+9\right)dx \\ \Rightarrow \frac{1}{3}du &= \left(4x^3+3\right)dx \end{aligned}$$

3. Analyze in terms of $u$ and $du$

$$\int \underbrace{\left(3x^4+9x\right)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}}\underbrace{\left(4x^3+3\right)dx}_{\frac{1}{3}du} = \int u^{-\frac{1}{2}}\frac{1}{3}du = \frac{1}{3}\int u^{-\frac{1}{2}}du$$

4. Integrate (in terms of u).

$$\frac{1}{3}\int u^{-\frac{1}{2}}du = \frac{1}{3}\left[\frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right] + C = \frac{2}{3}u^{\frac{1}{2}} + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{(4x^3+3)}{\sqrt{3x^4+9x}}dx = \underbrace{\frac{2}{3}\left(3x^4+9x\right)^{\frac{1}{2}} + C}_{\frac{2}{3}u^{\frac{1}{2}}+C}$$

i.e., $\int \frac{(4x^3+3)}{\sqrt{3x^4+9x}}dx = \frac{2}{3}\left(3x^4+9x\right)^{\frac{1}{2}} + C$

5. Compute: $\int \sec^2\left(2x^2+1\right)\, x\, dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $\sec^2\left(2x^2+1\right)$

      outer ↗ ↖ inner

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = 2x^2+1$

   b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(2x^2+1\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = 2x^2+1$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of $u$ ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute $du$

$$\begin{aligned} u &= 2x^2+1 \\ \Rightarrow \frac{du}{dx} &= 4x \\ \Rightarrow du &= 4x\, dx \\ \Rightarrow \frac{1}{4}du &= x\, dx \end{aligned}$$

3. Analyze in terms of $u$ and $du$

$$\int \underbrace{\sec^2\left(2x^2+1\right)}_{\sec^2(u)} \underbrace{x\, dx}_{\frac{1}{4}du} = \int \sec^2(u)\ \frac{1}{4}du = \frac{1}{4}\int \sec^2(u)\, du$$

4. Integrate (in terms of u).

$$\frac{1}{4}\int \sec^2(u)\, du = \frac{1}{4}\left[\tan(u)\right] + C = \frac{1}{4}\tan(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \sec^2\left(2x^2+1\right)\, x\, dx = \underbrace{\frac{1}{4}\tan\left(2x^2+1\right)+C}_{\frac{1}{4}\tan(u)+C}$$

$$\text{i.e., } \int \sec^2\left(2x^2+1\right)\, x\, dx = \frac{1}{4}\tan\left(2x^2+1\right)+C$$

6. Compute: $\int \frac{\sec^2(x)}{\tan(x)} dx =$

$$\int \frac{\sec^2(x)}{\tan(x)} dx \underbrace{=}_{\text{re-write}} \int \frac{1}{\tan(x)} \sec^2(x)\ dx$$

**Remark:** Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{\tan(x)}$ is the same as $(\tan(x))^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \tan(x)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\tan(x)}_{\text{function}} - - - - \rightarrow \underbrace{\sec^2(x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \tan(x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of $u$ ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute $du$

$$\begin{aligned} u &= \tan(x) \\ \Rightarrow \frac{du}{dx} &= \sec^2(x) \\ \Rightarrow du &= \sec^2(x)\, dx \end{aligned}$$

3. Analyze in terms of $u$ and $du$

$$\int \underbrace{\int \frac{1}{\tan(x)}}_{\frac{1}{u}} \underbrace{\sec^2(x)\ dx}_{du} = \int \frac{1}{u} du$$

4. Integrate (in terms of u).

$$\int \frac{1}{u} du = \ln|u| + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{\sec^2(x)}{\tan(x)} dx = \underbrace{\ln|\tan(x)| + C}_{\ln|u|+C}$$

$$\text{i.e., } \int \frac{\sec^2(x)}{\tan(x)} dx = \ln|\tan(x)| + C$$

7. Compute: $\frac{d}{dx}\left[\ln\left(\cot(x)+3\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\cot(x)+3\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\cot(x)+3}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(-\csc^2(x)\right)}_{g'(x)} = -\frac{\csc^2(x)}{\cot(x)+3}$$

i.e., $\frac{d}{dx}\left[\ln\left(\cot(x)+3\right)\right] = -\frac{\csc^2(x)}{\cot(x)+3}$

8. Compute: $\frac{d}{dx}\left[\ln\left(2x^2+6x-3\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(2x^2+6x-3\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{2x^2+6x-3}}_{\frac{1}{g(x)}} \cdot \underbrace{(4x+6)}_{g'(x)} = \frac{4x+6}{2x^2+6x-3}$$

i.e., $\frac{d}{dx}\left[\ln\left(2x^2+6x-3\right)\right] = \frac{4x+6}{2x^2+6x-3}$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sin(x)\sqrt{x^2-1}\right)\right] = \frac{d}{dx}\left[\ln\left(\sin(x)\left(x^2-1\right)^{\frac{1}{2}}\right)\right]$

**Remark:** We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left(\sin(x)\left(x^2-1\right)^{\frac{1}{2}}\right)\right] = \underbrace{\frac{d}{dx}\left[\ln(\sin(x)) + \ln\left(\left(x^2-1\right)^{\frac{1}{2}}\right)\right]}_{\ln(ab)=\ln(a)+\ln(b)} = \underbrace{\frac{d}{dx}\left[\ln(\sin(x)) + \frac{1}{2}\ln\left(x^2-1\right)\right]}_{\ln a^n = n\ln(a)}$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(\sin(x)\left(x^2-1\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\ln(\sin(x)) + \frac{1}{2}\ln\left(x^2-1\right)\right] = \frac{1}{\sin(x)}\cos(x) + \frac{1}{2}\frac{1}{x^2-1}(2x)$$

$$= \frac{\cos(x)}{\sin(x)} + \frac{x}{x^2-1} = \cot(x) + \frac{x}{x^2-1}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sin(x)\sqrt{x^2-1}\right)\right] = \frac{\cos(x)}{\sin(x)} + \frac{x}{x^2-1} = \cot(x) + \frac{x}{x^2-1}$

10. Compute: $\int_{x=0}^{x=2} \sqrt{x^3+1}\, x^2\, dx \underbrace{=}_{\text{re-write}} \int_{x=0}^{x=2} \left(x^3+1\right)^{\frac{1}{2}} x^2\, dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^3+1\right)^{\frac{1}{2}}$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^3+1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^3+1\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x^2}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^3+1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of $u$ ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute $du$

$$\begin{aligned} u &= x^3+1 \\ \Rightarrow \frac{du}{dx} &= 3x^2 \\ \Rightarrow du &= 3x^2\, dx \\ \Rightarrow \frac{1}{3}du &= x^2\, dx \end{aligned}$$

When $x=0$, $u = x^3+1 = (0)^3+1 = 1$
When $x=2$, $u = x^3+1 = (2)^3+1 = 9$

3. Analyze in terms of $u$ and $du$

$$\int_{x=0}^{x=2} \underbrace{\left(x^3+1\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{x^2\, dx}_{\frac{1}{3}du} = \int_{u=1}^{u=9} u^{\frac{1}{2}} \cdot \frac{1}{3}du = \frac{1}{3}\int_{u=1}^{u=9} u^{\frac{1}{2}}\, du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\frac{1}{3}\int_{u=1}^{u=9} u^{\frac{1}{2}}\, du = \frac{1}{3}\left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_{u=1}^{u=9} = \frac{2}{9}\left[u^{\frac{3}{2}}\right]_{u=1}^{u=9} = \underbrace{\frac{2}{9}(9)^{\frac{3}{2}}}_{F(9)} - \underbrace{\frac{2}{9}(1)^{\frac{3}{2}}}_{F(1)} = \frac{2}{9}(27) - \frac{2}{9}(1) = \frac{52}{9}$$

i.e., $\int_{x=0}^{x=2} \sqrt{x^3+1}\, x^2\, dx = \frac{52}{9}$