MTH 1126 - Practice Test #2_1 - Solutions

FALL 2015

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Instructions. Show CLEARLY how you arrive at your answers.

1. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = 2x - 1$.

   First, graph the functions and find the points of intersection.

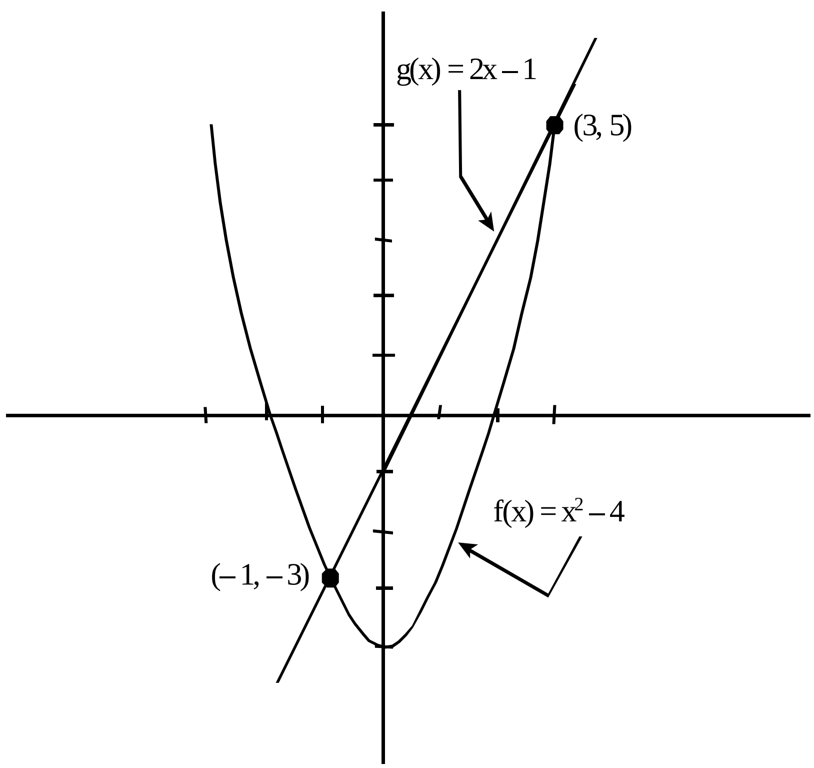
   $y = x^2 - 4 = 2x - 1$

   $\Rightarrow x^2 - 2x - 3 = 0$

   $\Rightarrow (x+1)(x-3) = 0$

   $x = -1;\ x = 3$

   Points of intersection are $(-1, -3)$ and $(3, 5)$.



   The bounded region spans the interval $[-1, 3]$ on the x-axis. Over this interval, $g(x) = 2x - 1$ is greater than $f(x) = x^2 - 4$. Hence the area is given by:

   $\int_{-1}^{3} [(2x-1) - (x^2-4)]\,dx = \int_{-1}^{3} (-x^2 + 2x + 3)\,dx = \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3}$

   $= \left(-\frac{1}{3}(3)^3 + (3)^2 + 3(3)\right) - \left(-\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right) = \frac{32}{3}$

2. Suppose that $\int_2^8 f(x)\,dx = 9$ and that $\int_4^2 f(x)\,dx = 4$. Compute $\int_4^8 f(x)\,dx$.

   Observe: $\int_4^2 f(x)\,dx + \int_2^8 f(x)\,dx = \int_4^8 f(x)\,dx$

   Hence, $\int_4^8 f(x)\,dx = 9 + 4 = 13$

   Alternatively, $\int_2^4 f(x)\,dx + \int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx$

   Hence, $\int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx - \int_2^4 f(x)\,dx = \int_2^8 f(x)\,dx - \left(-\int_4^2 f(x)\,dx\right) = \int_2^8 f(x)\,dx + \int_4^2 f(x)\,dx = 9 + 4 = 13$

3. Find the area bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

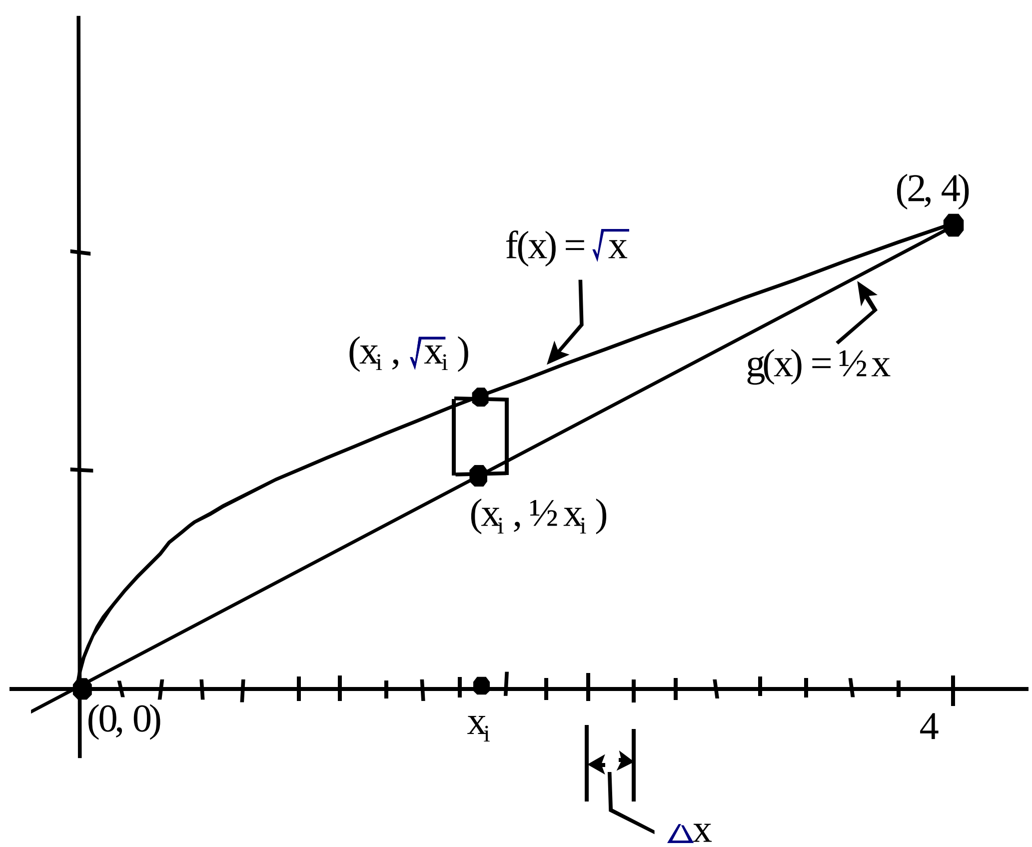
   Graph the functions and find the points of intersection.

   $y = \sqrt{x} = \frac{1}{2}x \Rightarrow x = \frac{1}{4}x^2 \Rightarrow \frac{1}{4}x^2 - x = 0 \Rightarrow x^2 - 4x = 0$

   $\Rightarrow x(x-4) = 0.$

   $\Rightarrow x = 0$; and $x = 4$.

   Points of intersection: $(0,0)$ and $(4,2)$.



The rectangles span the interval $[0, 4]$ on the x-axis, so we will partition that interval into sub-intervals of length Δx.

The area of the i^{th}. rectangle is $\underbrace{\left(\sqrt{x_i} - \frac{1}{2}x_i\right)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}$

The approximate area of the bounded region is given by:

$A \approx \sum_{i=1}^{n} \left(\sqrt{x_i} - \frac{1}{2}x_i\right) \Delta x$

To get the exact area, we let $\Delta x \to 0$.

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(\sqrt{x_i} - \frac{1}{2}x_i\right) \Delta x = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right) dx = \int_0^4 \left(x^{\frac{1}{2}} - \frac{1}{2}x\right) dx =$
$\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2\right]_0^4 = \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{4}(4)^2\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{4}(0)^2\right) = \frac{16}{3} - 4 = \frac{4}{3}$

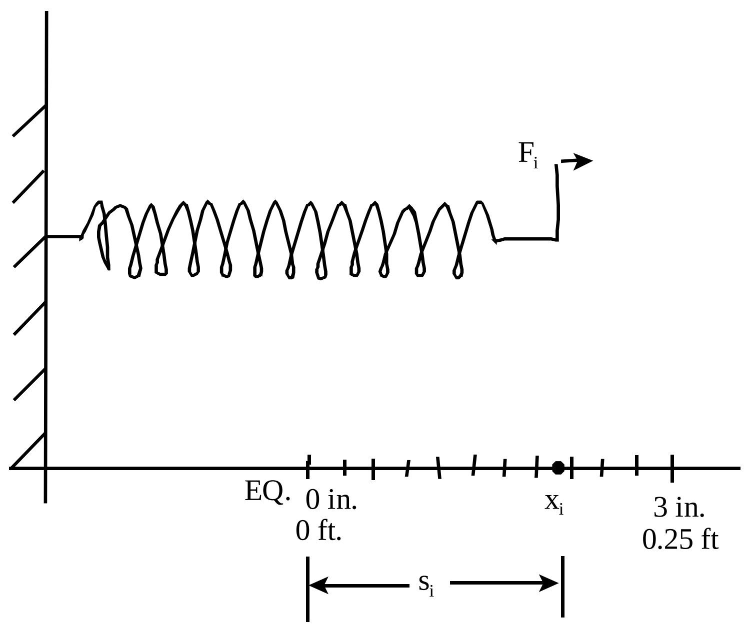
4. 10 pounds of force is required to stretch a spring 6 inches past the point of equilibrium. How much work is done stretching the spring 3 inches past the point of equilibrium? (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

First, find the spring constant, k, using the values $F = 10$ lb and $s = 6$ inches $= \frac{1}{2}$ ft

From Hooke's Law, $F = ks$, we have $k = \frac{F}{s} = \frac{10 \text{ lb}}{\frac{1}{2} \text{ ft}} = 20 \ \frac{\text{lb}}{\text{ft}}$

Hence, we have: $F = 20\frac{\text{lb}}{\text{ft}} s$

Next, partition the interval, over which the work is to be performed, and compute W_i, the work done stretching the spring from one side of the i^{th} sub-interval to the other side of the i^{th} sub-interval.



$W_i = F_i d_i$

Here, $d_i = \Delta x$

$F_i = ks_i = 20\frac{\text{lb}}{\text{ft}} x_i$

Hence, $W_i = 20\frac{\text{lb}}{\text{ft}} x_i \Delta x$

The total work, W_T, is approximately the sum of the work done stretching the spring across each sub-interval.

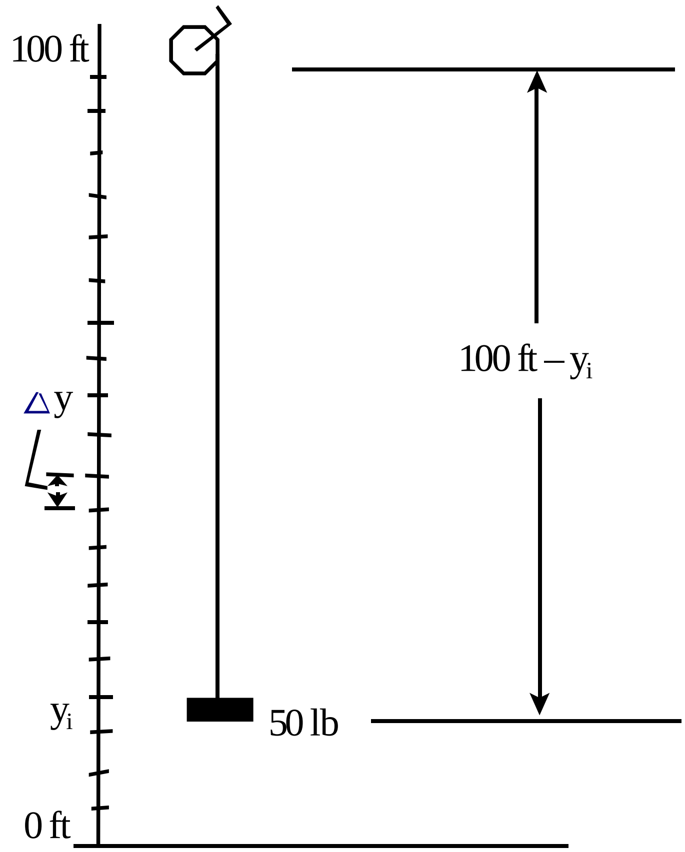
$W_T \approx \sum_{i=1}^{n} 20\frac{\text{lb}}{\text{ft}} x_i \Delta x$

$W_T = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 20\frac{\text{lb}}{\text{ft}} x_i \Delta x = \int_{0 \text{ ft}}^{0.25 \text{ ft}} 20\frac{\text{lb}}{\text{ft}} x \, dx = 20\frac{\text{lb}}{\text{ft}} \int_{0 \text{ ft}}^{0.25 \text{ ft}} x \, dx = 20\frac{\text{lb}}{\text{ft}} \left[\frac{x^2}{2}\right]_{0 \text{ ft}}^{0.25 \text{ ft}}$

$= 20\frac{\text{lb}}{\text{ft}} \left[\left(\frac{(0.25 \text{ ft})^2}{2}\right) - \left(\frac{(0 \text{ ft})^2}{2}\right)\right] = \frac{5}{8}$ lb ft

5. A cable, weighing 1 pound per foot length, is used to pull a 50 pound weight from ground level to a height of 100 feet, using a winch. How much work is done in the process? (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

Partition the interval over which the weight will travel, and compute W_i the work done raising the weight from the bottom to the top of the i^{th} sub-interval.



$W_i = F_i d_i$

Here, $d_i = \Delta y$, and F_i is the combined weight of the unwound portion of the cable plus the 50 lb weight itself.

$F_i = (\text{weight of cable}) + (50 \text{ lb weight})$

$F_i = (\text{length of cable})(\text{weight per unit length}) + (50 \text{ lb})$

$F_i = (100 \text{ ft } - y_i)\left(\frac{1 \text{ lb}}{\text{ft}}\right) + (50 \text{ lb})$

$F_i = \left(100 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) + (50 \text{ lb})$

$F_i = \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right)$

Hence, $W_i = F_i d_i = \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) \Delta y$

To compute the total work done, W_T, we add up the work done in raising the weight from bottom to top of each sub-interval.

$W_T \approx \sum_{i=1}^{n} \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) \Delta y$

To get the EXACT work done, we let $\Delta y \to 0$.

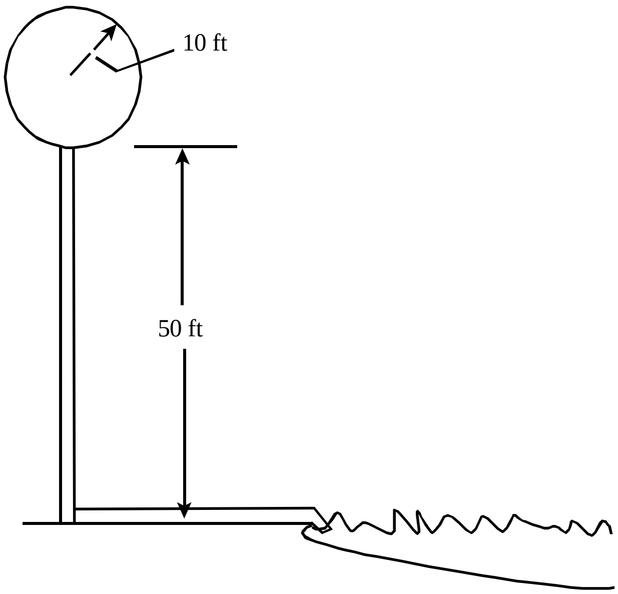
$W_T = \lim_{\Delta y \to 0} \sum_{i=1}^{n} \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) \Delta y$

$= \int_{0 \text{ ft}}^{100 \text{ ft}} \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y\right) dy = \left[150 \text{ lb } y - \frac{1 \text{ lb}}{\text{ft}} \frac{y^2}{2}\right]_{0 \text{ ft}}^{100 \text{ ft}}$

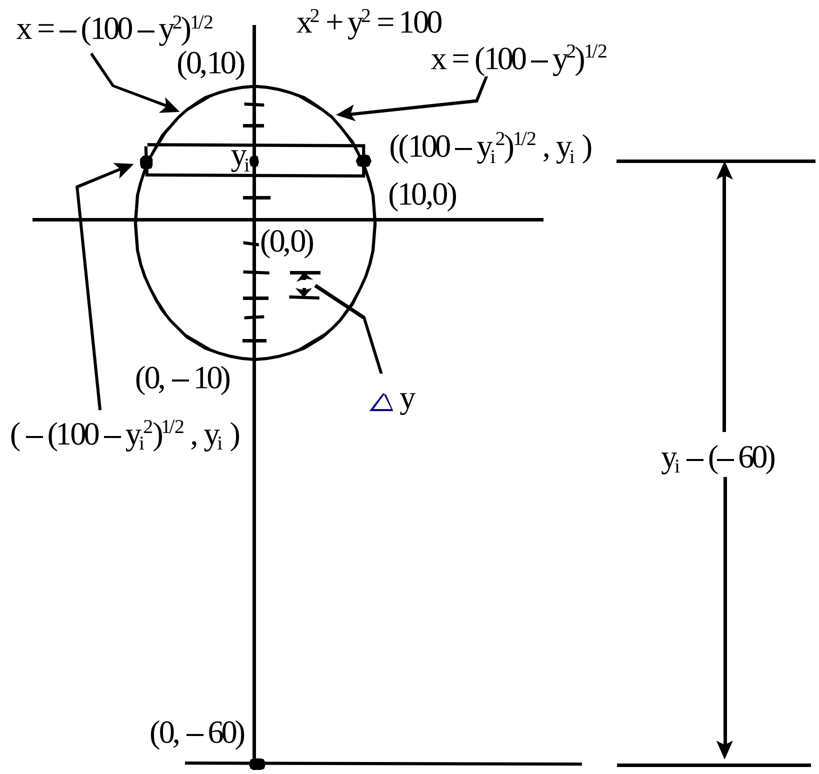
$= \left[\left(150 \text{ lb } (100 \text{ ft}) - \frac{1 \text{ lb}}{\text{ft}} \frac{(100 \text{ ft})^2}{2}\right) - \left(150 \text{ lb } (0 \text{ ft}) - \frac{1 \text{ lb}}{\text{ft}} \frac{(0 \text{ ft})^2}{2}\right)\right]$

$= 15,000 \text{ lb ft} - 5,000 \text{ lb ft} = 10,000 \text{ lb ft}$

6. A water tower has a spherical reservior of radius 10 feet. If the bottom of the reservior is 50 feet from ground level, how much work is done filling the reservior by pumping water into the reservior through a hole in the bottom? (Assume that water weighs 100 pounds per cubic foot.) (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)



Situate the tower in the x-y plane. It is probably best to situate the tower so that the center of the sphere coincides with the origin.



Partition the water into horizontal layers, of width Δy.

W_i is the work done pumping the ith layer of water to its final height.

$W_i = F_i d_i$

Here, d_i is the distance that the ith layer of water is pumped.

$d_i = y_i + 60$ ft

F_i is the weight of the ith layer.

$F_i = \left(\text{volume of i}^{th} \text{ layer}\right)(\text{weight per unit volume})$

$F_i = \left(\pi R_i^2 \Delta y\right) \rho$, where $\rho = 100 \ \frac{\text{lb}}{\text{ft}^3}$

$$F_i = \left(\pi \left(\sqrt{100 \text{ ft}^2 - y_i^2}\right)^2 \Delta y\right) \rho = \rho\pi \left(100 \text{ ft}^2 - y_i^2\right) \Delta y$$

Hence, $W_i = F_i d_i = \left(\rho\pi \left(100 \text{ ft}^2 - y_i^2\right) \Delta y\right) \left(y_i + 60 \text{ ft}\right)$

$$= \rho\pi \left(100 \text{ ft}^2 y_i - y_i^3 - 60 \text{ ft } y_i^2 + 6,000 \text{ ft}\right) \Delta y$$

i.e., $W_i = \rho\pi \left(100 \text{ ft}^2 y_i - y_i^3 - 60 \text{ ft } y_i^2 + 6,000 \text{ ft}\right) \Delta y$

Compute the total work done W_T, by computing the work done pumping each layer to its final height.

$W_T \approx \sum_{i=1}^{n} \rho\pi \left(100 \text{ ft}^2 y_i - y_i^3 - 60 \text{ ft } y_i^2 + 6,000 \text{ ft}\right) \Delta y$

Let $\Delta y \to 0$

$W_T = \lim_{\Delta y \to 0} \sum_{i=1}^{n} \rho\pi \left(100 \text{ ft}^2 y_i - y_i^3 - 60 \text{ ft } y_i^2 + 6,000 \text{ ft}\right) \Delta y$

$$= \rho\pi \int_{-10 \text{ ft}}^{10 \text{ ft}} \left(100 \text{ ft}^2 y - y^3 - 60 \text{ ft } y^2 + 6,000 \text{ ft}\right) dy$$

$$= \rho\pi \left[50 \text{ ft}^2 y^2 - \tfrac{1}{4} y^4 - 20 \text{ ft } y^3 + 6,000 \text{ ft } y\right]_{-10 \text{ ft}}^{10 \text{ ft}}$$

$$= \rho\pi \left(50 \text{ ft}^2 \left(10 \text{ ft}\right)^2 - \tfrac{1}{4} \left(10 \text{ ft}\right)^4 - 20 \text{ ft } \left(10 \text{ ft}\right)^3 + 6,000 \text{ ft } \left(10 \text{ ft}\right)\right)$$

$$-\rho\pi \left(50 \text{ ft}^2 \left(-10 \text{ ft}\right)^2 - \tfrac{1}{4} \left(-10 \text{ ft}\right)^4 - 20 \text{ ft } \left(-10 \text{ ft}\right)^3 + 6,000 \text{ ft } \left(-10 \text{ ft}\right)\right)$$

$$= \rho\pi \left(80,000 \text{ ft}^4\right) = 100 \tfrac{\text{lb}}{\text{ft}^3} \pi \left(80,000 \text{ ft}^4\right) = 8,000,000 \ \pi \text{ lb ft}$$

7. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = 2x^2 - 4$ and $g(x) = x^2$.

First, graph the functions and find the points of intersection.

To find the points of intersection, set $f(x) = g(x)$.

$f(x) = 2x^2 - 4 = x^2 = g(x)$

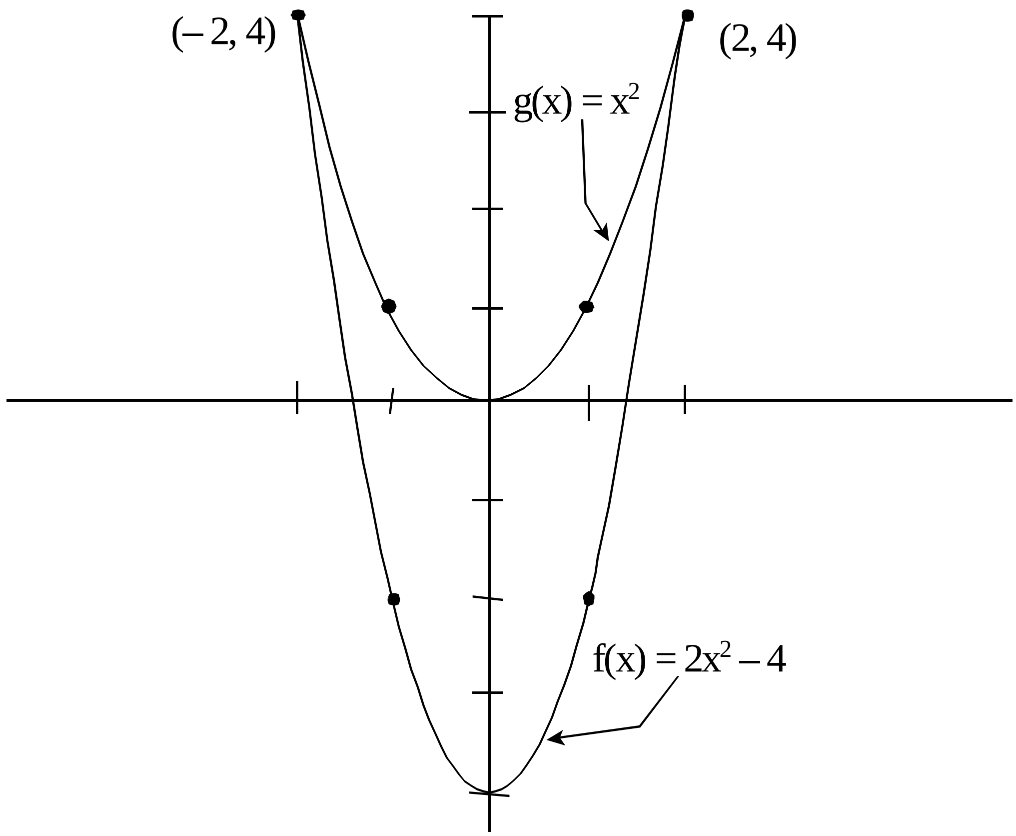
$\Rightarrow 2x^2 - 4 = x^2$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x+2)(x-2) = 0$

$\Rightarrow x = -2;\ x = 2$

The points of intersection are: $(-2, 4)$ and $(2, 4)$.



Since $x^2 \geq 2x^2 - 4$ over the interval $[-2, 2]$, the area of the bounded region is given by

$\int_{-2}^{2} [(x^2) - (2x^2 - 4)]\, dx = \int_{-2}^{2} (-x^2 + 4)\, dx = \left[-\frac{1}{3}x^3 + 4x\right]_{-2}^{2} =$

$\left(-\frac{1}{3}(2)^3 + 4(2)\right) - \left(-\frac{1}{3}(-2)^3 + 4(-2)\right) = \frac{32}{3}$

8. Find the area bounded by the graphs of $f(x) = x^2 - 2$ and $g(x) = 2x + 1$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

(a) 1. First, graph the functions and find the points of intersection.

To find the points of intersection, set $f(x) = g(x)$.

$f(x) = x^2 - 2 = 2x + 1 = g(x)$

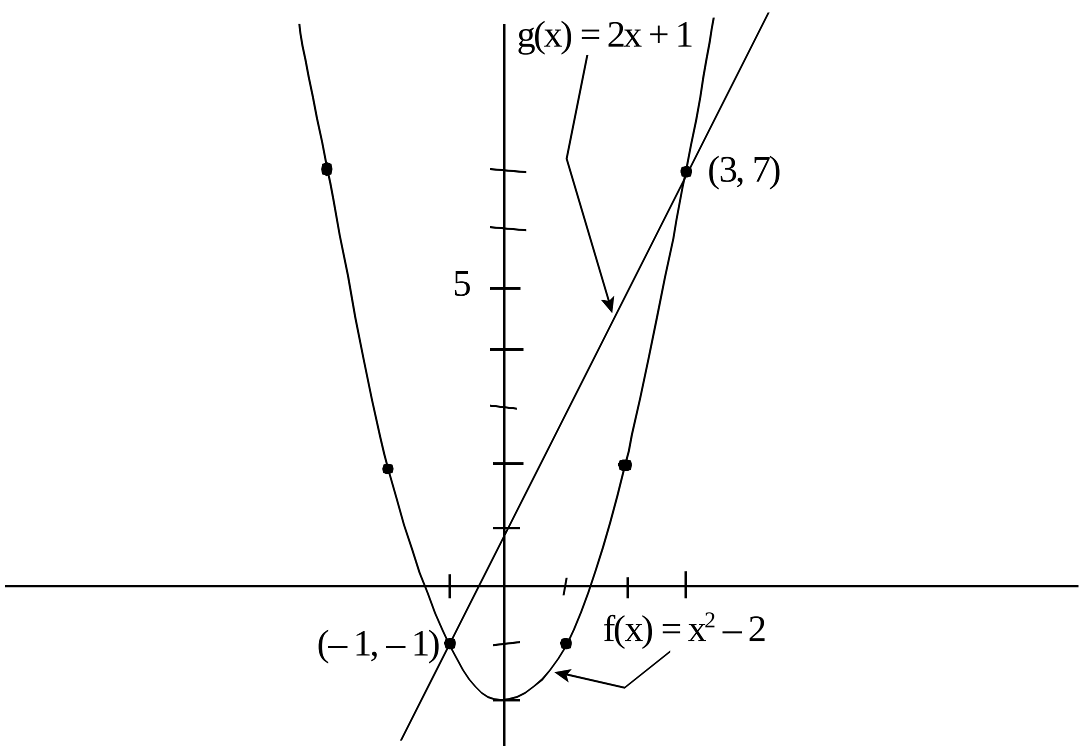
$\Rightarrow x^2 - 2 = 2x + 1$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$

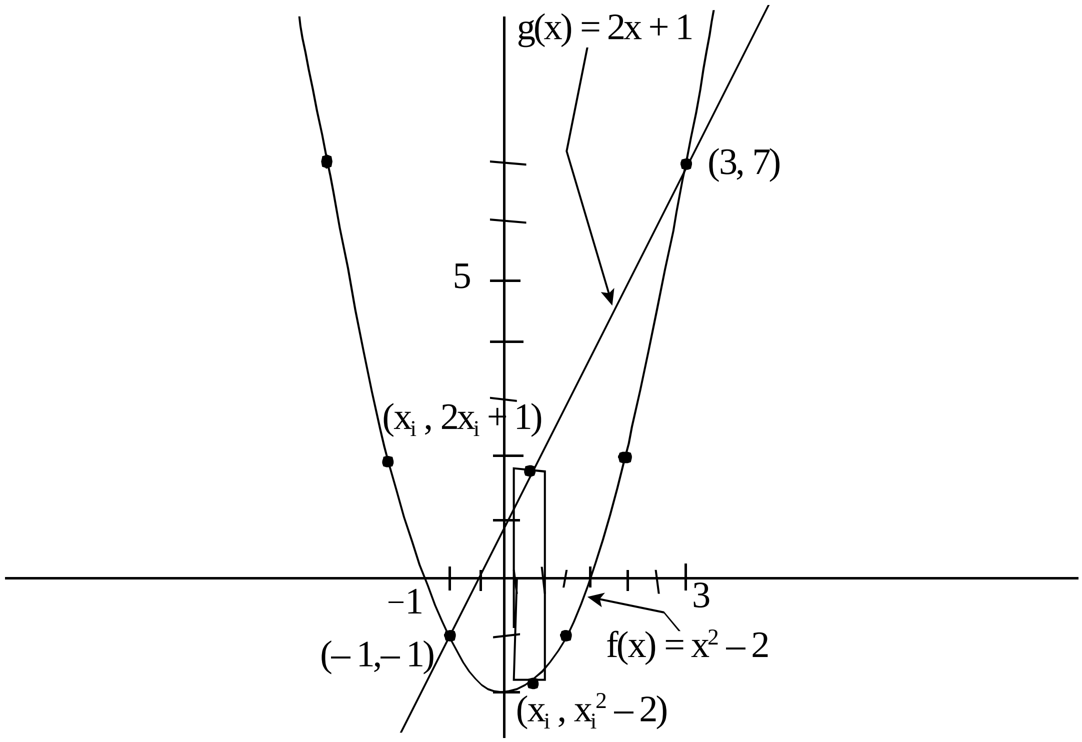
$\Rightarrow x = -1;\ x = 3$

Points of intersection are $(-1, -1)$ and $(3, 7)$.



2. Partition the interval spanned by the region into sub-intervals of length Δx.

3. Above the i^{th} subinterval, inscribe a rectangle of width Δx.



Height of the i^{th} rectangle $= [(2x_i + 1) - (x_i^2 - 2)] = 2x_i + 3 - x_i^2$

Width of the i^{th} rectangle $= \Delta x$

Area of the i^{th} rectangle $= (2x_i + 3 - x_i^2)\,\Delta x$

4. Approximate the area of the region by adding up the areas of the rectangles.

   Area $\approx \sum_{i=1}^{n} (2x_i + 3 - x_i^2)\,\Delta x$

5. Let $\Delta x \to 0$.

   Area $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} (2x_i + 3 - x_i^2)\,\Delta x = \int_{-1}^{3} (2x + 3 - x^2)\,dx = \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

   $= \left((3)^2 + 3\,(3) - \frac{(3)^3}{3}\right) - \left((-1)^2 + 3\,(-1) - \frac{(-1)^3}{3}\right) = \frac{32}{3}$

9. Use the "$f-g$" method to compute the area bounded by the graphs of $f(x)=x^3$ and $g(x)=4x$.

First, graph the functions and find the points of intersection.

To find the points of intersection, set $f(x)=g(x)$.

$f(x)=x^3=4x=g(x)$

$\Rightarrow x^3=4x$

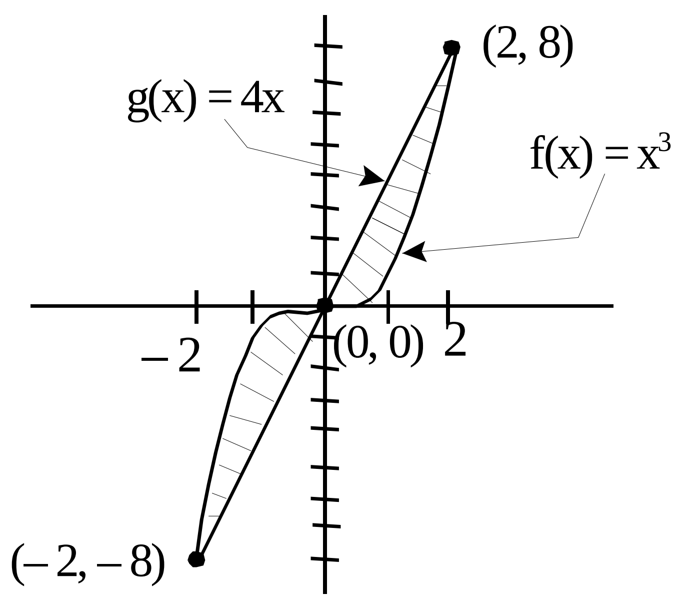
$\Rightarrow x^3-4x=0$

$\Rightarrow x(x^2-4)$

$\Rightarrow x(x+2)(x-2)=0$

$\Rightarrow x=0,\ x=-2;\ x=2$

The points of intersection are: $(-2,-8)$, $(0,0)$, and $(2,8)$.



Since the graphs of $y=x^3$ and $y=4x$ "flip flop" at $x=0$, we must compute the total bounded area *using two integrals.*

Total Bounded Area = (Area between x = – 2 and x = 0) + (Area between x = 0 and x = 2)

Since $4x \leq x^3$ over the interval $[-2,0]$, the area of the bounded region between x = – 2 and x = 0 is given by $\int_{-2}^{0}(x^3-4x)\,dx$.

Since $4x \geq x^3$ over the interval $[0,2]$, the area of the bounded region is given by $\int_0^2 (4x - x^3)\ dx$.

Total Bounded Area $= \int_{-2}^0 (x^3 - 4x)\ dx + \int_0^2 (4x - x^3)\ dx = \left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^0 + \left[2x^2 - \frac{1}{4}x^4\right]_0^2$

$= \left[\left(\frac{1}{4}(0)^4 - 2(0)^2\right) - \left(\frac{1}{4}(-2)^4 - 2(-2)^2\right) -\right] + \left[\left(2(2)^2 - \frac{1}{4}(2)^4\right) - \left(2(0)^2 - \frac{1}{4}(0)^4\right)\right]$

$= 8$

Total Bounded Area $= 8$

10. Suppose that $\int_2^8 (f(x) + g(x))\, dx = 10$; $\int_2^8 g(x)\, dx = 5$; and that $\int_4^2 f(x)\, dx = 4$. Compute $\int_4^8 f(x)\, dx$.

First, we need to find the value of $\int_2^8 f(x)\, dx$.

Since $\int_2^8 (f(x) + g(x))\, dx = \int_2^8 f(x)\, dx + \int_2^8 g(x)\, dx$, it follows that

$\int_2^8 f(x)\, dx = \int_2^8 (f(x) + g(x))\, dx - \int_2^8 g(x)\, dx = 10 - 5 = 5$

i.e., $\int_2^8 f(x)\, dx = 5$

Next observe that $\int_4^2 f(x)\, dx + \int_2^8 f(x)\, dx = \underbrace{\int_4^8 f(x)\, dx}_{\text{this is what we want}}$

Thus, $\int_4^8 f(x)\, dx = \int_4^2 f(x)\, dx + \int_2^8 f(x)\, dx = 4 + 5 = 9$

i.e., $\int_4^8 f(x)\, dx = 9$

11. Find the area bounded by the graphs of $f(x) = 1 - x^2$ and $g(x) = x^2 - 1$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

(a) 1. First, graph the functions and find the points of intersection.

To find the points of intersection, set $f(x) = g(x)$.

$f(x) = 1 - x^2 = x^2 - 1 = g(x)$

$\Rightarrow 1 - x^2 = x^2 - 1$

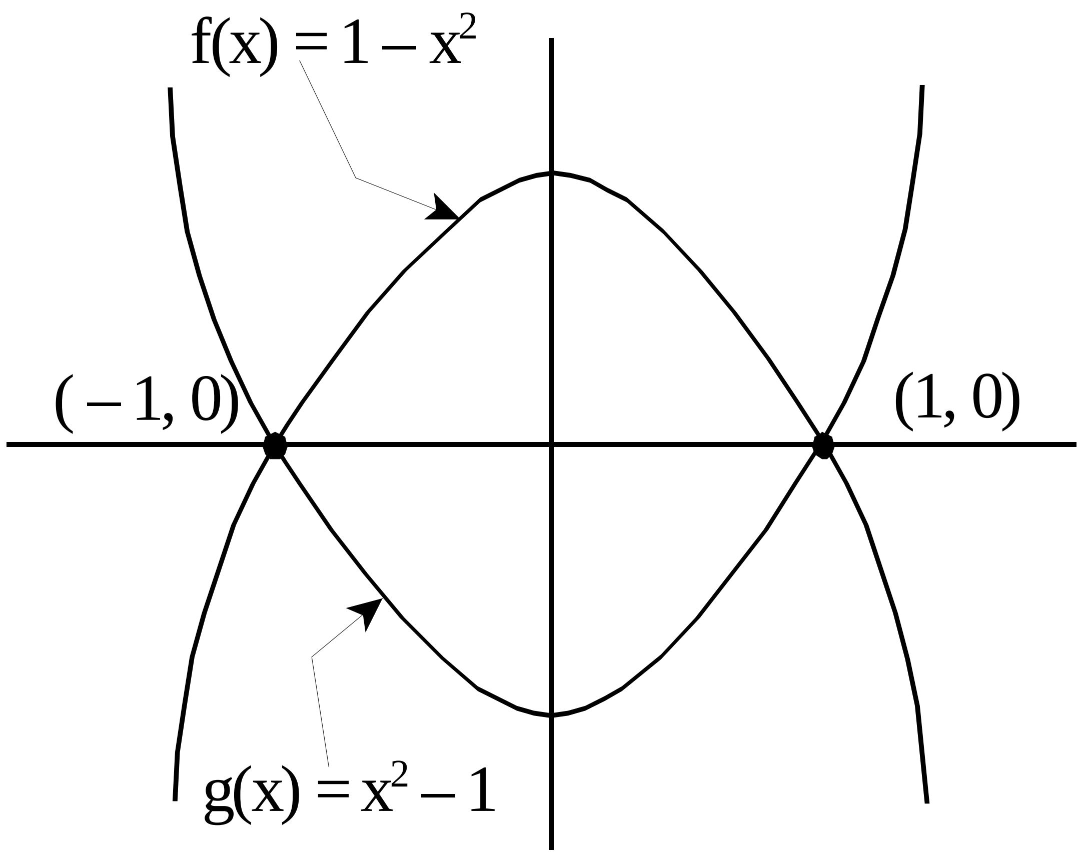
$\Rightarrow 2 - 2x^2 = 0$

$\Rightarrow 1 - x^2 = 0$

$\Rightarrow (x + 1)(x - 1) = 0$

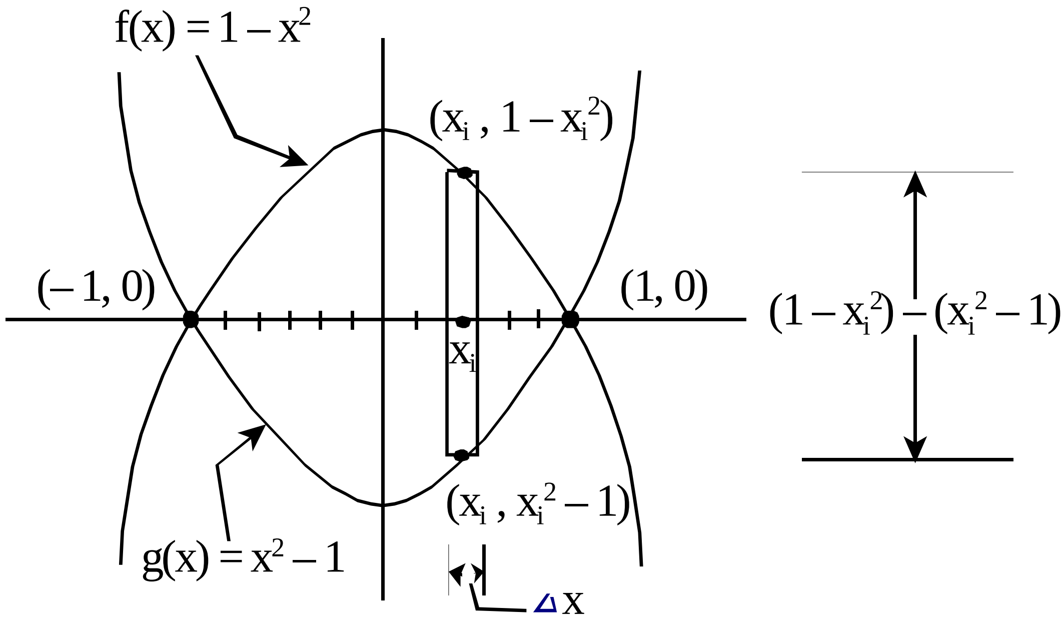
$\Rightarrow x = -1;\ x = 1$

Points of intersection are $(-1, 0)$ and $(1, 0)$.



2. Partition the interval spanned by the region into sub-intervals of length Δx.

3. Above the i^{th} subinterval, inscribe a rectangle of width Δx.



Height of the i^{th} rectangle $= [(1 - x_i^2) - (x_i^2 - 1)] = 2 - 2x_i^2$

Width of the i^{th} rectangle $= \Delta x$

Area of the i^{th} rectangle $= (2 - 2x_i^2)\,\Delta x$

4. Approximate the area of the region by adding up the areas of the rectangles.

   Area $\approx \sum_{i=1}^{n} (2 - 2x_i^2)\,\Delta x$

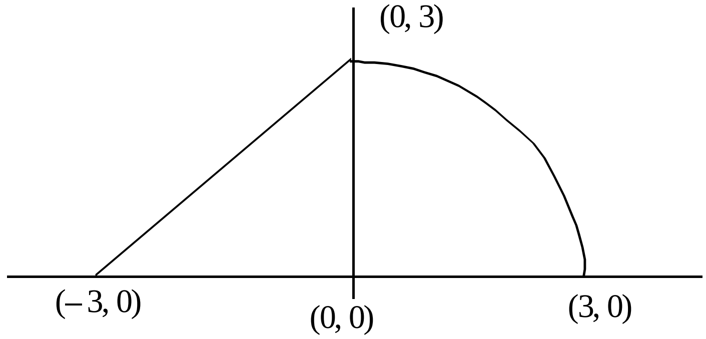
5. Let $\Delta x \to 0$.

   Area $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} (2 - 2x_i^2)\,\Delta x = \int_{-1}^{3} (2 - 2x^2)\,dx = \left[2x - \frac{2}{3}x^3\right]_{-1}^{1}$

   $= \left(2\,(1) - \frac{2}{3}\,(1)^3\right) - \left(2\,(-1) - \frac{2}{3}\,(-1)^3\right) = \frac{8}{3}$

   i.e., Bounded area $= \frac{8}{3}$

12. The graph of $f(x)$ is shown below. Compute $\int_{-3}^{3} f(x)\,dx$



Observe that between $x = -3$ and $x = 3$, $f(x) \geq 0$. Therefore, $\int_{-3}^{3} f(x)\,dx$ is equal to the area bounded by the graph of $f(x)$ and the x-axis, between $x = -3$ and $x = 3$.

Thus, $\int_{-3}^{3} f(x)\,dx = (\text{area of triangle with base and height} = 3) + \left(\text{area of } \frac{1}{4} \text{ circle of radius } 3\right)$

$= \frac{1}{2} b \cdot h + \frac{1}{4}\pi r^2 = \frac{1}{2}(3)(3) + \frac{1}{4}\pi(3)^2 = \frac{9}{2} + \frac{9\pi}{4} = \frac{18+9\pi}{4}$

13. Use the "$f-g$" method to compute the area bounded by the graphs of the functions $f(x)=x^2+2$; and $g(x)=x+8$.

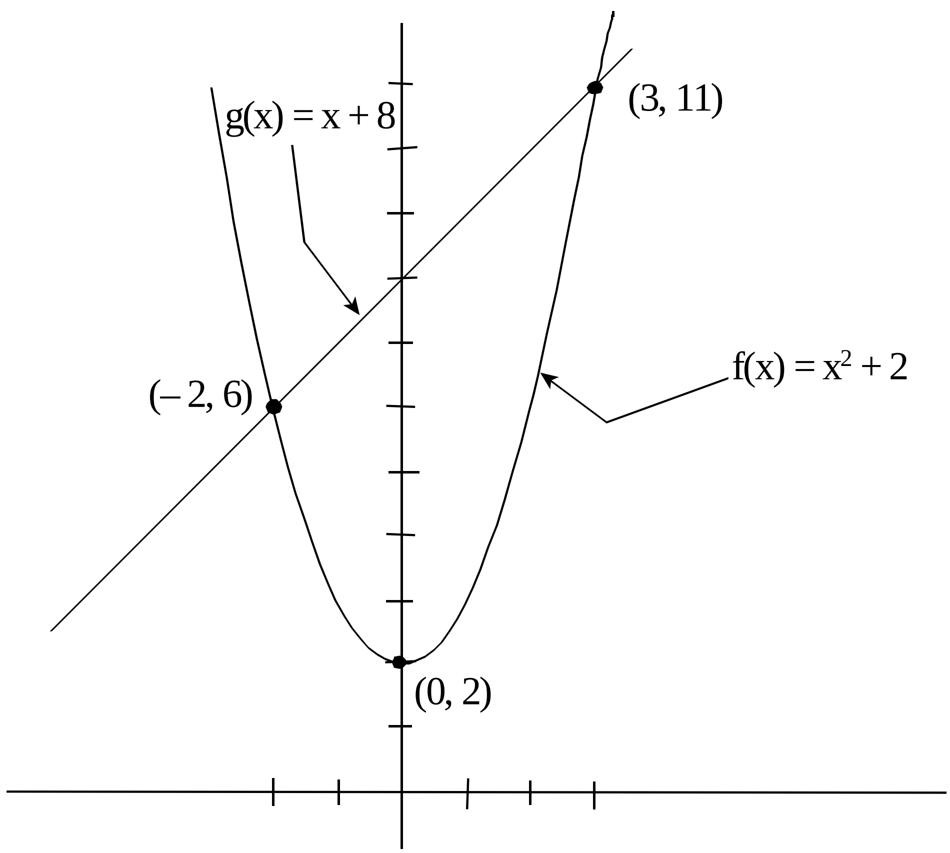
First, we graph the functions. The points of intersection are found by setting $f(x)=g(x)$.

$\Rightarrow f(x)=x^2+2=x+8=g(x)$

$\Rightarrow x^2-x-6=0$

$\Rightarrow (x+2)(x-3)=0$

$\Rightarrow x=-2$; $x=3$ are the points of intersection. (i.e., $(-2,6)$ and $(3,11)$)



As you can see, $g(x)=x+8\geq f(x)=x^2+2$ over the interval $[-2,3]$.

Hence, the bounded area is given by

$\int_{-2}^{3}[(x+8)-(x^2+2)]\,dx=\int_{-2}^{3}(x+6-x^2)\,dx=\left[\frac{x^2}{2}+6x-\frac{x^3}{3}\right]_{-2}^{3}$

$=\left(\frac{(3)^2}{2}+6(3)-\frac{(3)^3}{3}\right)-\left(\frac{(-2)^2}{2}+6(-2)-\frac{(-2)^3}{3}\right)=\frac{125}{6}$

i.e., area is $\frac{125}{6}$.

14. Compute the arclength of the graph of the function $f(x) = x^{\frac{3}{2}} + 6$ from the point $(1,7)$ and $(4,14)$.

Use the formula: Arc Length $= \int_a^b \sqrt{1+(f'(x))^2}dx$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$(f'(x))^2 = \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2 = \frac{9}{4}x$

$\Rightarrow$Arc Length $= \int_a^b \sqrt{1+(f'(x))^2}dx = \int_{x=1}^{x=4} \sqrt{1+\frac{9}{4}x}dx = \int_{x=1}^{x=4} \underbrace{\left(1+\frac{9}{4}x\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{\frac{4}{9}du}$

$$\boxed{\begin{array}{rcl} u &=& 1+\frac{9}{4}x \\ \Rightarrow\quad du &=& \frac{9}{4}dx \\ \Rightarrow\quad \frac{4}{9}du &=& dx \\ && \text{When } x=1,\ u = 1+\frac{9}{4}(1) = \frac{13}{4} \\ && \text{When } x=4,\ u = 1+\frac{9}{4}(4) = 10 \end{array}}$$

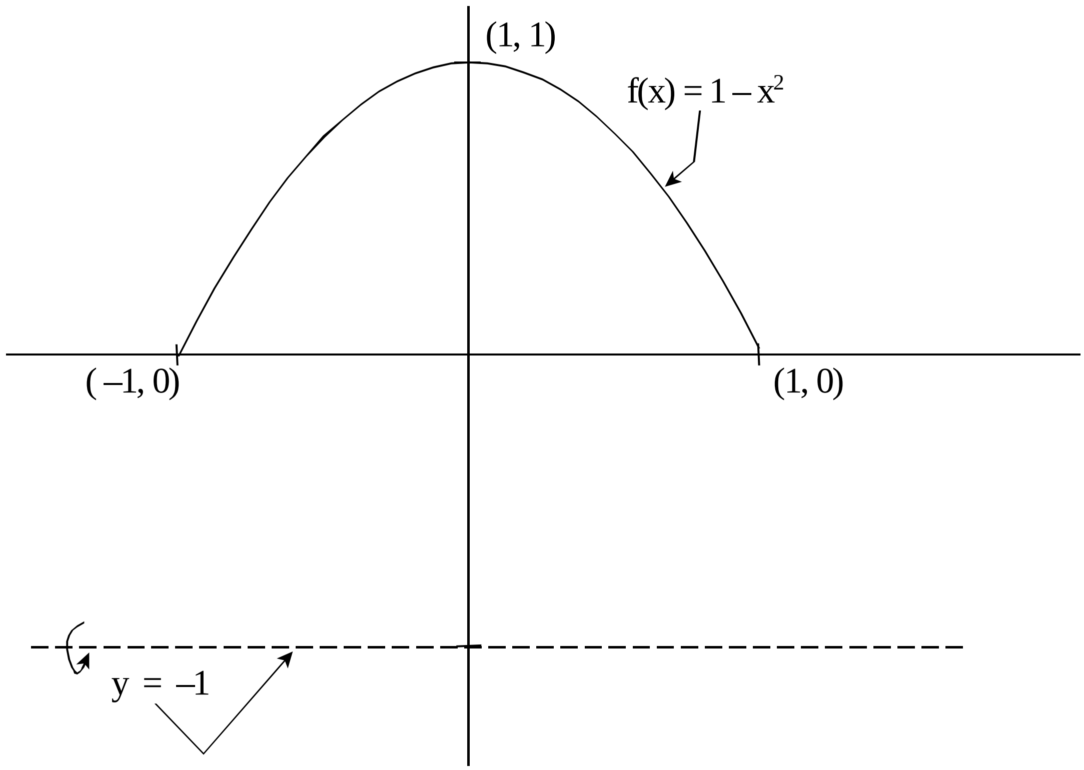
$= \int_{u=\frac{13}{4}}^{u=10} u^{\frac{1}{2}}\frac{4}{9}du = \frac{4}{9}\int_{u=\frac{13}{4}}^{u=10} u^{\frac{1}{2}}du = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=\frac{13}{4}}^{u=10} = \frac{8}{27}(10)^{\frac{3}{2}} - \frac{8}{27}\left(\frac{13}{4}\right)^{\frac{3}{2}}$

i.e., Arclength $= \frac{8}{27}(10)^{\frac{3}{2}} - \frac{8}{27}\left(\frac{13}{4}\right)^{\frac{3}{2}}$

For problems 15 to 16, use the “five step method” (partition the interval, form the sum, take the limit)

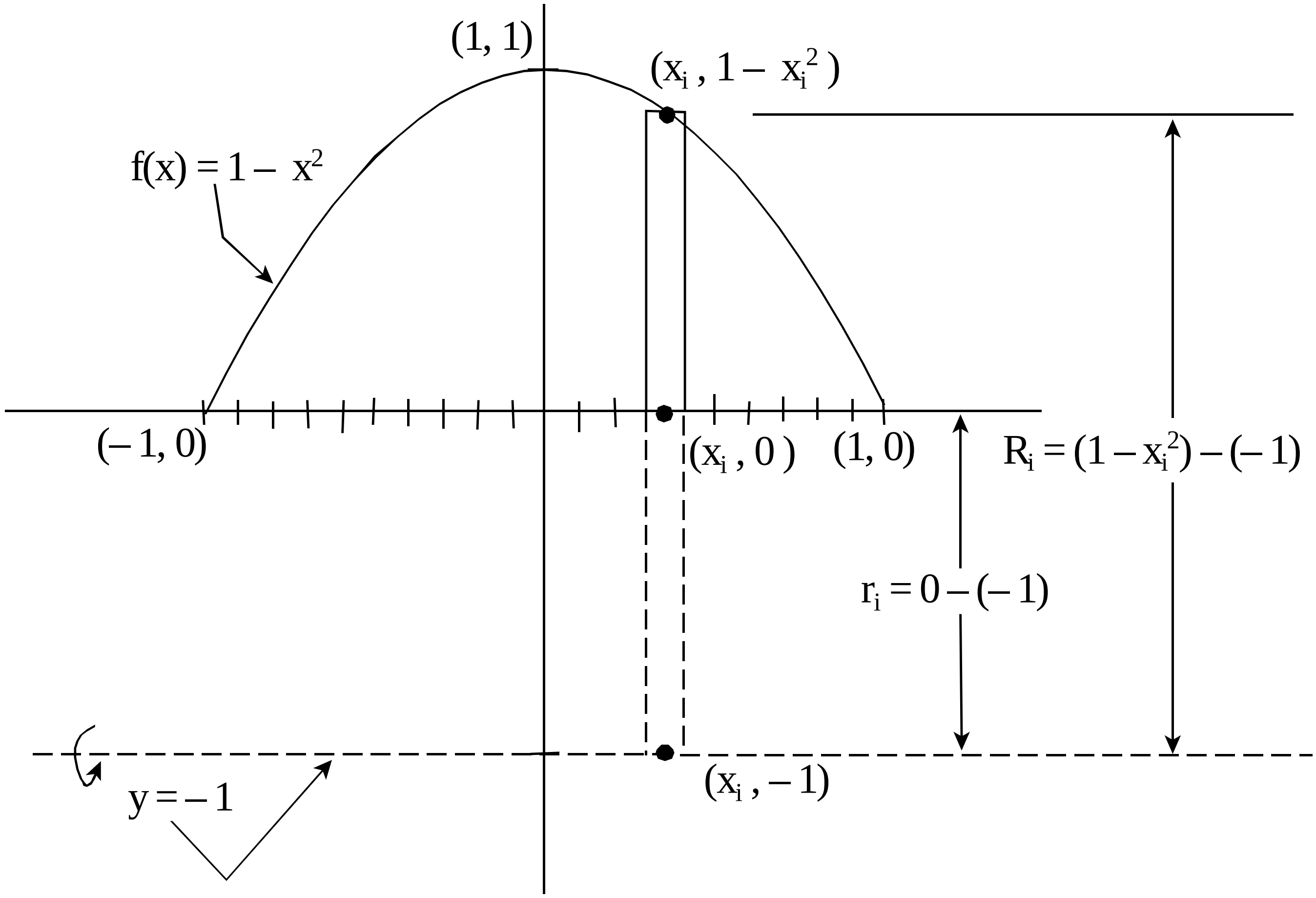
15. Use the “disc method” to compute the volume of the solid of revolution generated by revolving the region bounded by the graph $f(x) = 1 - x^2$ and the x-axis about the line $y = -1$.

(a) 1. First, graph the bounded area.



2. Sketch a rectangle perpendicular (perpen-“disc”-ular) to the axis of revolution

Partition the interval spanned by the rectangles.



3. Revolve the i[th] rectangle about the axis of revolution.

   Vol. of i[th] large disc $= \pi R_i^2 \Delta x = \pi \left(2 - x_i^2\right)^2 \Delta x = \pi \left(x_i^4 - 4x_i^2 + 4\right) \Delta x$

   Vol. of i[th] small disc $= \pi r_i^2 \Delta x = \pi \left(1\right)^2 \Delta x = \pi \Delta x$

   Vol. of i[th] washer $= \pi \left(x_i^4 - 4x_i^2 + 4\right) \Delta x - \pi \Delta x = \pi \left(x_i^4 - 4x_i^2 + 3\right) \Delta x$

4. Approximate the volume of the solid of revolution by adding up the volumes of the washers

   $Vol \approx \sum_{i=1}^{n} \pi \left(x_i^4 - 4x_i^2 + 3\right) \Delta x$

5. Let $\Delta x \to 0$

   $Vol \approx \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi \left(x_i^4 - 4x_i^2 + 3\right) \Delta x = \int_{x=-1}^{x=1} \pi \left(x^4 - 4x^2 + 3\right) dx$

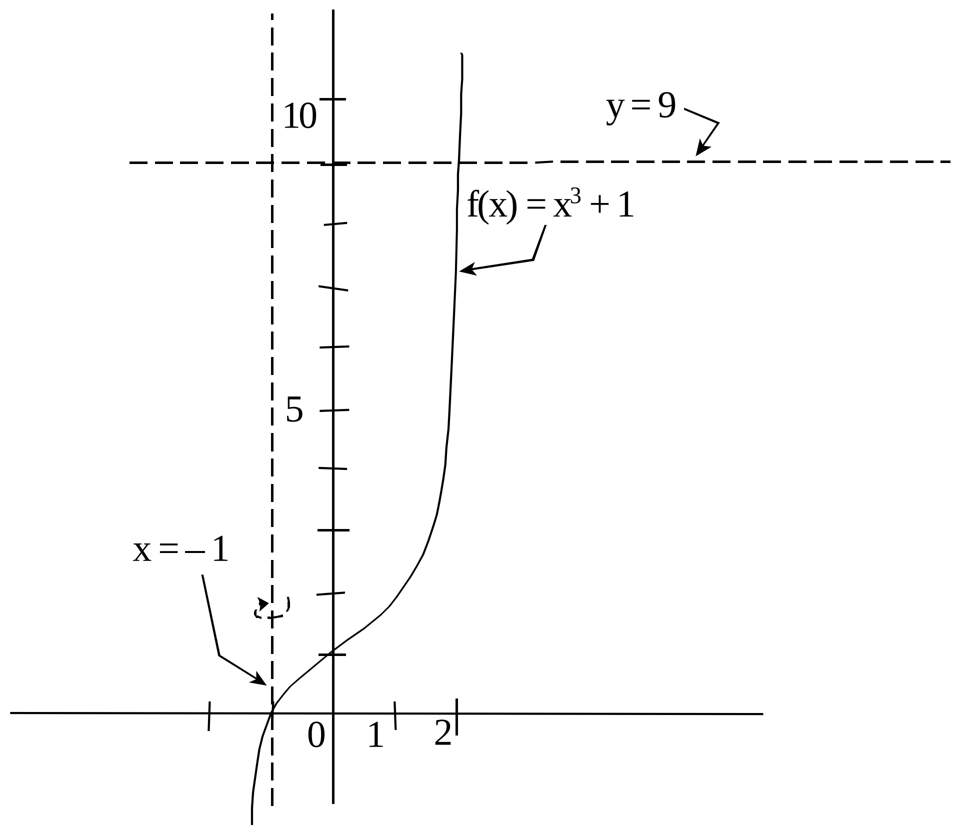
   $= \pi \left[\frac{x^5}{5} - 4\frac{x^3}{3} + 3x\right]_{x=-1}^{x=1} = \pi \left(\frac{(1)^5}{5} - 4\frac{(1)^3}{3} + 3\left(1\right)\right) - \pi \left(\frac{(-1)^5}{5} - 4\frac{(-1)^3}{3} + 3\left(-1\right)\right)$

   $= \frac{56}{15}\pi$

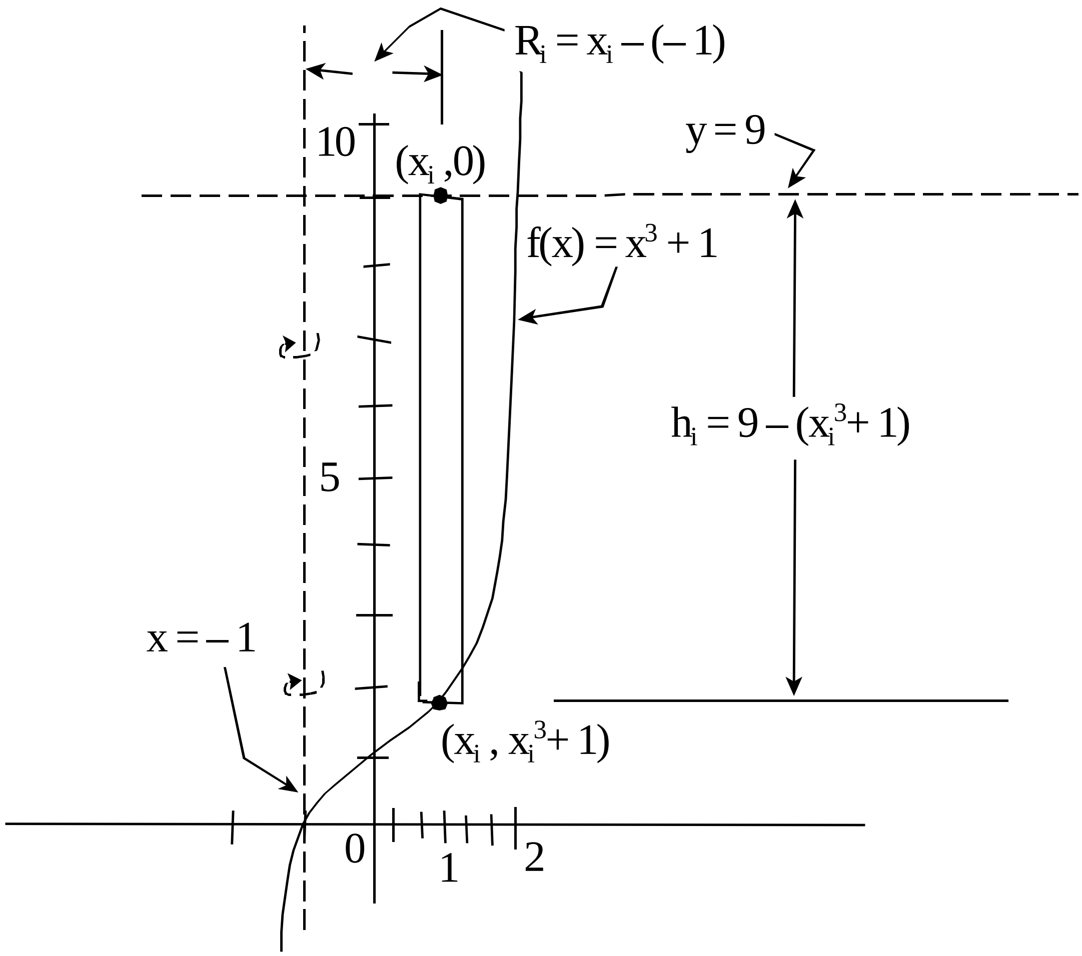
   i.e., Volume $= \frac{56\pi}{15}$

16. Use the “shell method” to compute the volume of the solid of revolution generated by revolving the region bounded by the graph $f(x) = x^3 + 1$, the y-axis, and the line $y = 9$, about the line $x = -1$.

(a) 1. First, graph the bounded area.



2. Sketch a rectangle parallel to the axis of revolution (“shell - parallel”), and partition the interval spanned by the rectangles



3. Revolve the i[th] rectangle about the axis of revolution to form the i[th] shell.

   Vol. i[th] shell $= 2\pi R_i h_i \Delta x = 2\pi\left(x_i+1\right)\left(8-x_i^3\right)\Delta x = 2\pi\left(8x_i+8-x_i^4-x_i^3\right)\Delta x$

4. Approximate the volume of the solid of revolution by adding the volumes of the shells.

   $Vol \approx \sum_{i=1}^{n} 2\pi\left(8x_i+8-x_i^4-x_i^3\right)\Delta x$

5. Let $\Delta x \to 0$

   $Vol = \lim_{\Delta x\to 0}\sum_{i=1}^{n} 2\pi\left(8x_i+8-x_i^4-x_i^3\right)\Delta x = \int_{x=0}^{x=2} 2\pi\left(8x+8-x^4-x^3\right)dx$

   $= 2\pi\left[4x^2+8x-\frac{x^5}{5}-\frac{x^4}{4}\right]_{x=0}^{x=2}$

   $= 2\pi\left(4\left(2\right)^2+8\left(2\right)-\frac{(2)^5}{5}-\frac{(2)^4}{4}\right)-2\pi\left(4\left(0\right)^2+8\left(0\right)-\frac{(0)^5}{5}-\frac{(0)^4}{4}\right) = \frac{216\pi}{5}$

   i.e., $Vol = \frac{216\pi}{5}$

17. $\int_0^5 f(x)\,dx = 8$ and $\int_5^3 2f(x)\,dx = 8$. Compute $\int_0^3 f(x)\,dx$.

We'd like to use the "formula": $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.

i.e., $\int_0^5 f(x)\,dx + \int_5^3 f(x)\,dx = \int_0^3 f(x)\,dx$.

The problem is, we don't have $\int_5^3 f(x)\,dx$, we have $\int_5^3 2f(x)\,dx$.

So our first order of business is to compute $\int_5^3 f(x)\,dx$.

Observe: $8 = \int_5^3 2f(x)\,dx = 2\int_5^3 f(x)\,dx$

i.e., $8 = 2\int_5^3 f(x)\,dx$

$\Rightarrow 4 = \int_5^3 f(x)\,dx$

i.e., $\int_5^3 f(x)\,dx = 4$.

Hence, se have $\int_0^3 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^3 f(x)\,dx = 8 + 4 = 12$

i.e., $\int_0^3 f(x)\,dx = 12$

18. Given that $x = \frac{y^3}{3} + \frac{1}{4y}$; compute the length of the arc from the point $\left(\frac{7}{12}, 1\right)$ to the point $\left(\frac{67}{24}, 2\right)$

Remark: This is the type of exercises where we expect the contents of the radical to be a perfect square.

$f(y) = \frac{1}{3}y^3 + \frac{1}{4}y^{-1}$

$f'(y) = y^2 - \frac{1}{4}y^{-2} = y^2 - \frac{1}{4y^2}$

$[f'(y)]^2 = \left(y^2 - \frac{1}{4y^2}\right)^2 = \left(y^4 - \frac{1}{2} + \frac{1}{16y^4}\right)$

$$\sqrt{1 + [f'(y)]^2} = \sqrt{1 + \left(y^4 - \frac{1}{2} + \frac{1}{16y^4}\right)} = \sqrt{\left(y^4 + \frac{1}{2} + \frac{1}{16y^4}\right)} = \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2} = \left(y^2 + \frac{1}{4y^2}\right)$$

$$\text{Arc Length} = \int_{y=1}^{y=2} \sqrt{1 + [f'(y)]^2}dy = \int_1^2 \left(y^2 + \frac{1}{4y^2}\right) dy = \int_1^2 \left(y^2 + \frac{1}{4}y^{-2}\right) dy = \left[\frac{1}{3}y^3 - \frac{1}{4}y^{-1}\right]_1^2$$

$$= \left[\frac{1}{3}y^3 - \frac{1}{4y}\right]_1^2 = \left[\frac{1}{3}(2)^3 - \frac{1}{4(2)}\right] - \left[\frac{1}{3}(1)^3 - \frac{1}{4(1)}\right] = \frac{59}{24}$$

i.e., arclength $= \frac{59}{24}$