Countability Problems

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Exercises

1. Prove: The set of even natural numbers, $\mathbf{E} = \{2, 4, 6, 8, \ldots, 2n, \ldots\}$, is countable.

   Proof. Observe:
   $$\begin{array}{rclcccccccl} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots & \} \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbf{E} & = & \{ & 2, & 4, & 6, & 8, & 10, & 12, & \ldots & \} \end{array}$$

   Define $f : \mathbf{N} \to \mathbf{E}$ by $f(n) = 2n$

   Clearly from the diagram above, f is one to one and onto. Hence, $\mathbf{E}$ is denumerable. ■

2. The set of integers that are multiples of 5, $5\mathbf{Z} = \{0, \pm 5, \pm 10, \pm 15, \ldots, \pm 5z, \ldots\}$, is countable.

   Proof. Observe:
   $$\begin{array}{rclcccccccccl} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots & \} \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ n\mathbf{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \ldots & \} \end{array}$$

   Define $f : \mathbf{N} \to \mathbf{5Z}$ by $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

   Clearly from the diagram above, f is one to one and onto. Hence, $5\mathbf{Z}$ is denumerable. ■

3. Prove: The set of irrational numbers, $\mathbf{Q}^c$, is uncountable.

   Remark: From the previous set of problems, we know that the union of two countably infinite (denumerable) sets is countably infinite (denumerable).

   Proof. (By contradiction)

   Suppose, for the sake of contradiction, that $\mathbf{Q}^c$ is countable (denumerable).

   Then $\mathbf{R} = \mathbf{Q} \cup \mathbf{Q}^c$ is the union of two denumerable sets, hence, denumerable.

   This contradicts the fact that $\mathbf{R}$ is uncountable (non-denumerable).

   Since the assumption that $\mathbf{Q}^c$ is countable (denumerable) leads to a contradiction, $\mathbf{Q}^c$ must be uncountable (non-denumerable). ■