MTH 4441 Exercises To study for Test #1 - Solutions

Fall 2023

Pat Rossi Name ____________________

1. In each case below, determine whether $*$ is a **closed** binary operation on the given set. If it *IS* a **closed** binary operation, then determine whether it is commutative and/or associative.

   (a) $(\mathbb{Z}, *)$ where $a * b = a + b^2$

   IS a **closed** binary operation on $\mathbb{Z}$

   NOT commutative. Given $a \neq b$, $a * b = a + b^2 \neq b + a^2 = b * a$

   NOT associative.

   $(a * b) * c = (a + b^2) * c = (a + b^2) + c^2 = a + b^2 + c^2$

   $a * (b * c) = a * (b + c^2) = a + (b + c^2)^2 = a + b^2 + 2bc^2 + c^4$

   $(a * b) * c \neq a * (b * c)$

   (b) $(\mathbb{Z}, *)$ where $a * b = a^2 b^3$

   IS a **closed** binary operation on $\mathbb{Z}$

   NOT commutative. Given $a \neq b$, $a * b = a^2 b^3 \neq b^2 a^3 = b * a$

   NOT associative.

   $(a * b) * c = a^2 b^3 * c = (a^2 b^3)^2 c^3 = a^4 b^6 c^3$

   $a * (b * c) = a * b^2 c^3 = a^2 (b^2 c^3)^3 = a^2 b^6 c^9$

   $(a * b) * c \neq a * (b * c)$

   (c) $(\mathbb{R}, *)$ where $a * b = \frac{a}{a^2 + b^2}$

   NOT a **closed** binary operation on $\mathbb{R}$ (e.g., $0 * 0$ is undefined, and hence, $0 * 0$ is not assigned **any** element. So $*$ is not even a binary operation!)

   (d) $(\mathbb{Z}, *)$ where $a * b = \frac{a^2 + 2ab + b^2}{a + b}$

   NOT a **closed** binary operation on $\mathbb{Z}$ (e.g., $0 * 0$ is undefined, and hence, $0 * 0$ is not assigned **any** element. So $*$ is not even a binary operation!)

(e) $(\mathbb{Z}, *)$ where $a * b = a + b - ab$

IS a **closed** binary operation on $\mathbb{Z}$

IS Commutative: $a * b = a + b - ab = b + a - ba = b * a$

i.e., $a * b = b * a$

IS Associative:

$$(a * b) * c = (a + b - ab) * c = (a + b - ab) + c - (a + b - ab)\,c$$

$$= (a + b - ab) + c - ac - bc + abc = a + b + c - ab - ac - bc + abc$$

$$a * (b * c) = a * (b + c - bc) = a + (b + c - bc) - a\,(b + c - bc)$$

$$= a + b + c - bc - ab - ac + abc = a + b + c - ab - ac - bc + abc$$

i.e., $(a * b) * c = a * (b * c)$

(f) $(\mathbb{R}, *)$ where $a * b = b$

IS a **closed** binary operation on $\mathbb{R}$

Is NOT Commutative: Given $a \neq b$, $a * b = b \neq a = b * a$

i.e., $a * b \neq b * a$

IS Associative:

$(a * b) * c = b * c = c$

$a * (b * c) = a * c = c$

i.e., $(a * b) * c = a * (b * c)$

(g) $(S, *)$ where $S = \{-4, -2, 1, 2, 3\}$ and $a * b = |b|$

Is NOT a **closed** binary operation on S. (e.g., $1 * (-4) = |4| = 4 \notin S$, and hence, $1 * (-4)$ is not closed on the set S)

(h) $(S, *)$ where $S = \{1, 2, 3, 6, 18\}$ and $a * b = ab$

NOT a **closed** binary operation on S. ($6 * 6 = 36 \notin S$. Hence. the operation is not closed on the set S.)

(i) $(S,*)$ where $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$ and $*$ is matrix addition

IS a **closed** binary operation on S

IS Commutative: Given $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, we have:

$$A + B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} = \begin{bmatrix} b_1+a_1 & b_2+a_2 \\ b_3+a_3 & b_4+a_4 \end{bmatrix} = B + A$$

i.e., $A + B = B + A$

IS Associative: Given $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$; $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$, we have:

$$(A+B)+C = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+(b_1+c_1) & a_2+(b_2+c_2) \\ a_3+(b_3+c_3) & a_4+(b_4+c_4) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{bmatrix}$$

$= A + (B + C)$

i.e., $(A+B)+C = A+(B+C)$

2. Let $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$, and let $(\mathbb{Z}_6, \oplus)$ be a group, where $\oplus$ is addition modulo 6. Construct the group table.

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

3. In the group $(\mathbb{Z}_6, \oplus)$, what is the order of the element 2? What is the order of the element 3?

(i.e., $o(2) = ?$ $\quad o(3) = ?$)

$1 \cdot 2 = 2;\ 2 \cdot 2 = 4;\ 3 \cdot 2 = 0; \quad o(2) = 3$

$1 \cdot 3 = 3;\ 2 \cdot 3 = 0; \quad o(3) = 2$

4. Construct the group table for $(\mathbb{Z}_7, \oplus)$.

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

5. Let $U_5 = \{1, 2, 3, 4\}$, and let $(U_5, \odot)$ be a group, where $\odot$ is multiplication modulo 5. Construct the group table.

   In $(U_5, \odot)$, the operation $\odot$ is multiplication modulo 5

$\odot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

6. Construct the group table for $(U_3, \odot)$.

   In $(U_3, \odot)$, the operation $\odot$ is multiplication modulo 3

$\odot$	1	2
1	1	2
2	2	1

7. Construct the group table for $(U_7, \odot)$.

   In $(U_7, \odot)$, the operation $\odot$ is multiplication modulo 7

$\odot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

8. Construct the group table for $(U_6, \odot)$.

$\odot$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

(a) $(U_6, \odot)$ is NOT a group. Give at least two reasons why it is not a group

1. U_6 is not closed under $\odot$. For example: $2 \odot 3 = 0$
2. None of the elements of U_6 appear at least once in every row and in every column. For example, 1 does not appear in the row headed by 2
3. The elements 2, 3, 4 appear more than once in some rows and columns. For example, 3 appears three times in the row and the column headed by 3.
4. The elements 2, 3, 4 do not have an inverse. This can be seen by the fact that the identity 1 does not appear in the rows and the columns that are headed by 2, 3, and 4.

9. Construct the group table for $(U_4, \odot)$.

In $(U_4, \odot)$, the operation $\odot$ is multiplication modulo 4

$\odot$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

(a) $(U_4, \odot)$ is NOT a group. Give at least two reasons why it is not a group

1. U_4 is not closed under $\odot$. For example: $2 \odot 2 = 0$
2. The elements 1 and 3 of U_4 do NOT appear at least once in every row and in every column. The elements 1 and 3 do not appear in the row or the column headed by 2.
3. The elements 2 appears more than once in the row and column headed by 2.
4. The elements 2 does not have an inverse. This can be seen by the fact that the identity 1 does not appear in the row and the column that are headed by 2.

10. Determine whether the table below defines a group for $G = \{a, b, c\}$. (State why or why not.)

$*$	a	b	c
a	a	b	c
b	b	a	c
c	c	b	a

Since the elements b and c each appear more than once in a column, the table does NOT define a group

Clearly, a is the identity. Therefore, there should not exist another element x such that $x * c = c$. And yet, $b * c = c$

Similarly, the element c is such that $c * b = b$

11. Determine whether the table below defines a group for $G = \{a, b, c\}$. (State why or why not.)

$*$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

This table does NOT define $(G, *)$ as a group.

1. The elements b and c appear more than once in the rows and columns headed by b and c.

2. The identity a does not appear in any row or column headed by b or c, which indicates that neither b nor c has an inverse.

12. Determine whether the table below defines a group for $G = \{a, b, c, d, e, f\}$. State why or why not. (You may assume that the operation $*$ is associative.)

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	d	f	a	c	e
c	c	f	b	e	a	d
d	d	a	e	b	f	c
e	e	c	a	f	d	b
f	f	e	d	c	b	a

This table **does** define a group.

The element a **is clearly the identity,** as $a * x = x$, and $x * a = x$. $\forall x \in G$

No elements outside of the set G appear in the table. This means that $x * y \in G$, $\forall x, y \in G$. Hence, **the set G is closed under** $*$.

Finally, the identity appears in each row and column, indicating that **each element of G has an inverse.** Furthermore, each right inverse is also the left inverse. (i.e., if $x * y = a$, then $y * x = a$)

13. In the previous exercise, what is the inverse of d? How do you know?

The inverse of d is b. The element a is the identity, and $b * d = a$ and $d * b = a$

14. Compute the remainder of 25 modulo 7 (i.e. $25 \equiv \_\_\_ \pmod 7$)

$25 = (3)(7) + 4$, Hence $25 \equiv \underline{4} \pmod 7$

15. Compute the remainder of 48 modulo 5 (i.e. $48 \equiv \_\_\_ \pmod 5$)

$48 = (9)(5) + 3$, Hence $48 \equiv \underline{3} \pmod 5$

16. Compute the remainder of 53 modulo 14 (i.e. $53 \equiv \_\_\_ \pmod{14}$)

$53 = (3)(14) + 11$, Hence $53 \equiv \underline{11} \pmod{14}$

17. Determine whether 58 and 75 are congruent modulo 9 (Determine whether $58 \equiv 75 \pmod 9$)

Method #1 $(58 - 75) = -17 \neq n(9)$, $\forall n \in \mathbb{Z}$. Hence $58 \not\equiv 75 \pmod 9$

Method #2 $58 = (6)(9) + 4$, and $75 = (8)(9) + 3$

Since 58 and 75 do not have the same remainder when divided by 9, $58 \not\equiv 75 \pmod 9$

18. Determine whether 43 and 59 are congruent modulo 16 (Determine whether $43 \equiv 59 \pmod 9$)

    Method #1 $(43 - 59) = -16 = (1)(16)$, $43 \equiv 59 \pmod{16}$

    Method #2 $43 = (2)(16) + 11$, and $59 = (3)(16) + 11$

    Since 43 and 59 have the same remainder when divided by 16, $43 \equiv 59 \pmod{16}$

19. Compute $\gcd(4, 18)$

    We will factor both into prime factors

    $4 = 2^2$ and $18 = 2 \cdot 3^2$

    Therefore, 2 is the greatest common divisor, or factor ($\gcd(4, 18) = 2$)

20. Compute $\gcd(25, 40)$

    We will factor both into prime factors

    $25 = 5^2$ and $40 = 2^3 \cdot 5$

    Therefore, 5 is the greatest common divisor, or factor ($\gcd(25, 40) = 5$)

21. Compute $\gcd(4, 25)$

    We will factor both into prime factors

    $4 = 2^2$ and $25 = 5^2$

    4 and 25 have no prime factors in common. Hence, $\gcd(4, 25) = 1$