MTH 1125 - Test 2 (12pm Class) - Solutions

Fall 2019

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[3x^5+4x^4+5x^3+8x^2+15x+32\sqrt{x}+2\right]=$

$\frac{d}{dx}\left[3x^5+4x^4+5x^3+8x^2+15x+32\sqrt{x}+2\right]$

$=3\left[5x^4\right]+4\left[4x^3\right]+5\left[3x^2\right]+8\left[2x\right]+15+32\left[\frac{1}{2}x^{-\frac{1}{2}}\right]+0$

$=15x^4+16x^3+15x^2+16x+15+16x^{-\frac{1}{2}}$

i.e., $\frac{d}{dx}\left[3x^5+4x^4+5x^3+8x^2+15x+32\sqrt{x}+2\right]=15x^4+16x^3+15x^2+16x+15+16x^{-\frac{1}{2}}$

2. Compute: $\frac{d}{dx}\left[\left(5x+\cos(x)\right)\left(x^4+3x^2\right)\right]=$

$$\frac{d}{dx}\left[\underbrace{\left(5x+\cos(x)\right)}_{1^{st}}\underbrace{\left(x^4+3x^2\right)}_{2^{nd}}\right]=\underbrace{\left(5-\sin(x)\right)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\left(x^4+3x^2\right)}_{2^{nd}}+\underbrace{\left(4x^3+6x\right)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\left(5x+\cos(x)\right)}_{1^{st}}$$

$\frac{d}{dx}\left[\left(5x+\cos(x)\right)\left(x^4+3x^2\right)\right]=\left(5-\sin(x)\right)\left(x^4+3x^2\right)+\left(4x^3+6x\right)\left(5x+\cos(x)\right)$

3. Compute: $\frac{d}{dx}\left[\frac{\sec(x)}{3x^2+6x+9}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{\sec(x)}^{\text{top}}}{\underbrace{3x^2+6x+9}_{\text{Bottom}}}\right]=\frac{\overbrace{\sec(x)\tan(x)}^{\text{top prime}}\cdot\overbrace{\left(3x^2+6x+9\right)}^{\text{bottom}}-\overbrace{\left(6x+6\right)}^{\text{bottom prime}}\cdot\overbrace{\sec(x)}^{\text{top}}}{\underbrace{\left(3x^2+6x+9\right)^2}_{\text{bottom squared}}}$$

i.e., $\frac{d}{dx}\left[\frac{\sec(x)}{3x^2+6x+9}\right]=\frac{\sec(x)\tan(x)\left(3x^2+6x+9\right)-(6x+6)\sec(x)}{\left(3x^2+6x+9\right)^2}$

4. Compute: $\frac{d}{dx}\left[\left(2x^4+3x^3+10x\right)^8\right]=$

$$\frac{d}{dx}\left[\left(2x^4+3x^3+10x\right)^8\right]=\underbrace{8\left(2x^4+3x^3+10x\right)^7}_{\substack{\text{power rule}\\\text{as usual}}}\cdot\underbrace{\left(8x^3+9x^2+10\right)}_{\substack{\text{derivative}\\\text{of inner}}}$$

> i.e., $\frac{d}{dx}\left[\left(2x^4+3x^3+10x\right)^8\right]=8\left(2x^4+3x^3+10x\right)^7\left(8x^3+9x^2+10\right)$

5. Given that $f(x)=3x^2+4x+5$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(1,12)$.

We need two things:

i. A point on the line (We have that: $(x_1,y_1)=(1,12)$)

ii. The slope of the line (This is $f'(x_1)$)

$f'(x)=6x+4$

At the point $(x_1,y_1)=(1,12)$, **the slope is** $f'(1)=6(1)+4=10$

We will use the Point-Slope equation of a line:

$(y-y_1)=m(x-x_1)$ (Where m is the slope and (x_1,y_1) is a known point on the line.)

Thus, the equation of the line tangent to the graph of $f(x)$ is:

$(y-12)=10(x-1)$

> The equation of the line tangent is $(y-12)=10(x-1)$

6. Given that $w = \tan(x)$ and that $x = v^2 + 2v + 3$; compute $\frac{dw}{dv}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

We know:

$\frac{dw}{dx} = \sec^2(x)$

$\frac{dx}{dv} = 2v + 2$

We want: $\frac{dw}{dv}$

By the Liebniz form of the Chain Rule:

$$\frac{dw}{dv} = \frac{dw}{dx}\frac{dx}{dv} = \sec^2(x)(2v+2) = \underbrace{\sec^2\left(v^2+2v+3\right)(2v+2)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } v}}$$

i.e. $\frac{dw}{dv} = \sec^2(v^2+2v+3)(2v+2)$

7. Compute: $\frac{d}{dx}\left[\cot\left(2x^3+4x^2+3\right)\right] =$

Outer:	=	$\cot(\quad)$
Deriv. of outer	=	$-\csc^2(\quad)$

$$\frac{d}{dx}\left[\begin{array}{c} \\ \cot\Big(\underbrace{2x^3+4x^2+3}\Big) \\ \uparrow \qquad\quad \uparrow \\ \text{outer} \quad \text{inner} \end{array}\right] = \underbrace{-\csc^2\left(2x^3+4x^2+3\right)}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\left(6x^2+8x\right)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\cot\left(2x^3+4x^2+3\right)\right] = -\csc^2\left(2x^3+4x^2+3\right)\left(6x^2+8x\right)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{3x^4+24x}{5x^3+12x}\right)^{10}\right] =$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE.

$$\frac{d}{dx}\left[\underbrace{\left(\frac{3x^4+24x}{5x^3+12x}\right)^{10}}_{(g(x))^n}\right] = \underbrace{10\left(\frac{3x^4+24x}{5x^3+12x}\right)^{9}}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{3x^4+24x}{5x^3+12x}\right]\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

$$= 10\left(\frac{3x^4+24x}{5x^3+12x}\right)^{9} \underbrace{\frac{(12x^3+24)(5x^3+12x) - (15x^2+12)(3x^4+24x)}{(5x^3+12x)^2}}_{\substack{\text{quotient}\\ \text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{3x^4+24x}{5x^3+12x}\right)^{10}\right] = 10\left(\frac{3x^4+24x}{5x^3+12x}\right)^{9} \cdot \frac{\left(12x^3+24\right)\left(5x^3+12x\right) - \left(15x^2+12\right)\left(3x^4+24x\right)}{(5x^3+12x)^2}$

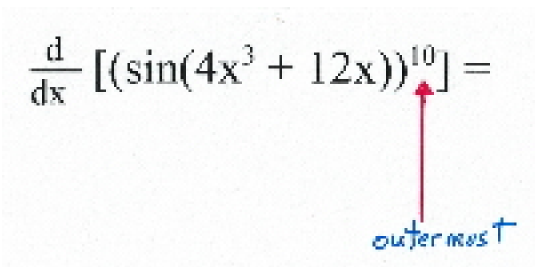
9. Compute: $\frac{d}{dx}\left[\sin^{10}\left(4x^3+12x\right)\right] =$

Let's rewrite this:

$\frac{d}{dx}\left[\left(\sin\left(4x^3+12x\right)\right)^{10}\right]$

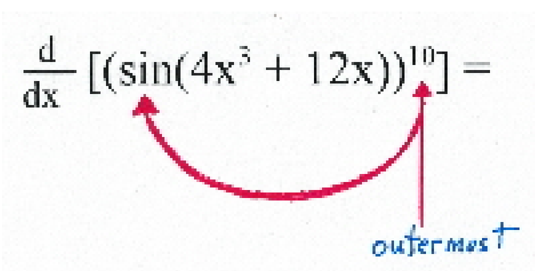
This is the composition of *three* functions.

Differentiate the outermost function and evaluate it at everything inside



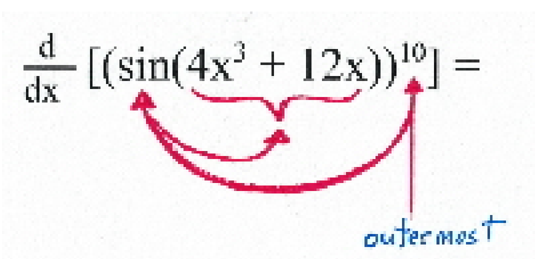
This yields: $10\left(\sin\left(4x^3+12x\right)\right)^9$

Next: Multiply by the derivative of the next outermost function and evaluate it at everything inside of it.



This yields: $10\left(\sin\left(4x^3+12x\right)\right)^9 \cdot \cos\left(4x^3+12x\right)$

Finally: Multiply by the derivative of the innermost function.



This yields: $10\left(\sin\left(4x^3+12x\right)\right)^9 \cos\left(4x^3+12x\right) \cdot \left(12x^2+12\right)$

i.e., $\frac{d}{dx}\left[\sin^{10}\left(4x^3+12x\right)\right] = 10\left(\sin\left(4x^3+12x\right)\right)^9 \cos\left(4x^3+12x\right)\left(12x^2+12\right)$

Alternatively:

$\frac{d}{dx}\left[\sin^{10}\left(4x^3+12x\right)\right]$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE. Re-write!

$\frac{d}{dx}\left[\left(\sin\left(4x^3+12x\right)\right)^{10}\right]$ This is the derivative of a function, raised to a power

$$\frac{d}{dx}\left[\left(\sin\left(4x^3+12x\right)\right)^{10}\right] = \underbrace{10\left(\sin\left(4x^3+12x\right)\right)^{9}}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\sin\left(4x^3+12x\right)\right]\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$= 10\left(\sin\left(4x^3+12x\right)\right)^{9} \cdot \underbrace{\left[\cos\left(4x^3+12x\right) \cdot \left(12x^2+12\right)\right]}_{\substack{\text{Chain}\\ \text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\sin\left(4x^3+12x\right)\right)^{10}\right] = 10\left(\sin\left(4x^3+12x\right)\right)^{9}\cos\left(4x^3+12x\right)\cdot\left(12x^2+12\right)$

10. Given that $x^3 + x^4y^5 = y^3$, compute y'

   i. Differentiate both sides w.r.t. x

   $$\frac{d}{dx}\left[x^3 + \underbrace{x^4}_{1^{\text{st}}}\underbrace{y^5}_{2^{\text{nd}}}\right] = \frac{d}{dx}\left[y^3\right]$$

   $$\Rightarrow 3x^2 + \left(\underbrace{4x^3}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^5}_{2^{\text{nd}}} + \underbrace{5y^4y'}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{x^4}_{1^{\text{st}}}\right) = 3y^2 \cdot y'$$

   ii. Solve algebraically for y'

   a. Get y' terms on left side, all other terms on right side

   $\Rightarrow 5y^4y'x^4 - 3y^2y' = -3x^2 - 4x^3y^5$

   b. Factor out y'

   $\Rightarrow \left(5y^4x^4 - 3y^2\right)y' = -3x^2 - 4x^3y^5$

   c. Divide both sides by the cofactor of y'

   $y' = \frac{-3x^2-4x^3y^5}{5y^4x^4-3y^2} = -\frac{3x^2+4x^3y^5}{5y^4x^4-3y^2}$

$y' = -\frac{3x^2+4x^3y^5}{5y^4x^4-3y^2}$

11. Given that $f(x) = 6x^2 - 4x + 6$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[6(x+\Delta x)^2-4(x+\Delta x)+6\right]-\left[6x^2-4x+6\right]}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left[6\left(x^2+2x\Delta x+\Delta x^2\right)-4(x+\Delta x)+6\right]-\left[6x^2-4x+6\right]}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left[6x^2+12x\Delta x+6\Delta x^2-4x-4\Delta x+6\right]-\left[6x^2-4x+6\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{12x\Delta x+6\Delta x^2-4\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta x(12x+6\Delta x-4)}{\Delta x} = \lim_{\Delta x \to 0} (12x + 6\Delta x - 4) = 12x + 6(0) - 4 = 12x - 4$

i.e., $f'(x) = 12x - 4$

Extra (Wow! 10 Points)

Given that $T'(x) = \frac{1}{1+x^2}$ (i.e., $\frac{d}{dx}[T(x)] = \frac{1}{1+x^2}$); compute $\frac{d}{dx}[T(\sin(x))]$

Outer:	$=$	$T(\quad)$
Deriv. of outer	$=$	$\frac{1}{1+(\quad)^2}$

$$\frac{d}{dx}\Big[\underset{\uparrow \atop \text{outer}}{T}\Big(\underbrace{\sin(x)}_{\uparrow \atop \text{inner}}\Big)\Big] = \underbrace{\frac{1}{1+(\sin(x))^2}}_{\text{Deriv of outer, eval. at inner}} \cdot \underbrace{\cos(x)}_{\text{deriv. of inner}} = \frac{\cos(x)}{1+(\sin(x))^2}$$

i.e., $\frac{d}{dx}[T(\sin(x))] = \frac{\cos(x)}{1+(\sin(x))^2}$