MTH 4425 Final Exam

Spring 2021

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Definition 1 *The infinite series*

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

is called the Harmonic Series.

Thm - The Harmonic Series diverges.

Proof. Consider the sequence of partial sums:

$S_1=1$

$S_2=1+\frac{1}{2}$

$S_3=1+\frac{1}{2}+\frac{1}{3}$

$\vdots$

$S_N=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{N}$

$\vdots$

Observe that the sequence of partial sums $\{S_1,S_2,S_3,\ldots,S_N,\ldots\}$ is strictly monotone increasing. Now consider the subsequence of partial sums:

$$S_1=\underbrace{1}_{>\frac{1}{2}}>\frac{1}{2}$$

$$S_2=\underbrace{1}_{>\frac{1}{2}}+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}>\frac{2}{2}$$

$$S_4=\underbrace{1}_{>\frac{1}{2}}+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}>\frac{3}{2}$$

$$S_8=\underbrace{1}_{>\frac{1}{2}}+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}>\frac{4}{2}$$

$$S_{16}=\underbrace{1}_{>\frac{1}{2}}+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{>\frac{1}{2}}>\frac{5}{2}$$

What we observe from this is that $S_{2^K}>\frac{K+1}{2}$ for $K=0,1,2,3,\ldots$

Combining this observation with the fact that the sequence of partial sums is monotone increasing, we conclude that given $M \in \mathbf{R}$, $\exists N = N(M) \in \mathbf{N}$ (namely $N = 2^{2\lceil M \rceil}$), such that:

$$n > N = 2^{2\lceil M \rceil} \Rightarrow S_n > S_{2\lceil M \rceil} > \frac{2\lceil M \rceil + 1}{2} > \frac{2\lceil M \rceil}{2} = \lceil M \rceil \geq M$$

$$\text{i.e., } \exists N = N(M) \in \mathbf{N} \text{ such that } n > N \Rightarrow S_n > M$$

i.e., the sequence of partial sums diverges.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$ diverges. ■

Thm - The series $\sum_{n=1}^{\infty} a_n$ converges only if $\lim_{n\to\infty} a_n = 0$

(i.e., If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$)

Proof. Let the hypothesis be given. (i.e., Suppose that $\sum_{n=1}^{\infty} a_n$ converges.)

Then this implies that the sequence of partial sums $\{S_1, S_2, S_3, \ldots, S_N, \ldots\}$ converges.

In turn, this implies that the sequence of partial sums $\{S_1, S_2, S_3, \ldots, S_N, \ldots\}$ is a Cauchy Sequence.

Thus, given $\varepsilon > 0$, $\exists M = M(\varepsilon) \in \mathbf{N}$ such that $m, n > M \Rightarrow |S_m - S_n| < \varepsilon$

Letting $N = M + 1$, we have: $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $m, n \geq N \Rightarrow |S_m - S_n| < \varepsilon$

In particular, for $n \geq N$, we have: $|S_{n+1} - S_n| < \varepsilon$

$\Rightarrow |a_n| < \varepsilon$

i.e., given $\varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n \geq N \Rightarrow |a_n - 0| < \varepsilon$

Thus, $\lim_{n\to\infty} a_n = 0$ ■

Definition 2 *A series of the form*

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots$$

is a Geometric Series with ratio r.

Thm - The Geometric Series $\sum_{n=0}^{\infty} ar^n$

i) Converges and has sum equal to $\frac{a}{1-r}$ if $|r| < 1$,

ii) Diverges if $|r| \geq 1$

Proof. Consider the N^{th} partial sum, $S_N = a + ar + ar^2 + \ldots + ar^N$.

Also consider: $r \cdot S_N = ar + ar^2 + ar^3 + \ldots + ar^N + ar^{N+1}$.

Subtracting, we have:

$$\begin{array}{rrcl} & S_N & = & a + ar + ar^2 + ar^3 + \ldots + ar^N \\ - & r \cdot S_N & = & \quad\; ar + ar^2 + ar^3 + \ldots + ar^N + ar^{N+1} \\ \hline & (1-r)\, S_N & = & a \qquad\qquad\qquad\qquad\qquad\qquad - ar^{N+1} \end{array}$$

i.e., $(1-r)\, S_N = a - ar^{N+1}$

$\Rightarrow S_N = \frac{a - ar^{N+1}}{1-r}$

Suppose that $|r| < 1$.

Then $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a-ar^{N+1}}{1-r} = \lim_{N\to\infty} \left(\frac{a}{1-r} - \frac{ar^{N+1}}{1-r} \right)$

$$= \lim_{N\to\infty} \frac{a}{1-r} - \lim_{N\to\infty} \frac{ar^{N+1}}{1-r} = \frac{a}{1-r} - \lim_{N\to\infty} \frac{a}{1-r} r^{N+1}$$

$$= \frac{a}{1-r} - \frac{a}{1-r} \cdot \underbrace{\lim_{N\to\infty} r^{N+1}}_{=0 \text{ because } |r|<1} = \frac{a}{1-r} - \frac{a}{1-r} \cdot 0 = \frac{a}{1-r}$$

i.e., $\lim_{N\to\infty} S_N = \frac{a}{1-r}$ and consequently, the series converges and has sum $\frac{a}{1-r}$.

Suppose that $|r| > 1$.

Then $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a-ar^{N+1}}{1-r} = \ldots = \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{N\to\infty} r^{N+1}$

This expression is equal to ∞ if $r > 1$, and diverges if $r < -1$. Either way, the series diverges.

Suppose that $|r| = 1$.

If $r = 1$, then $\sum_{n=0}^{\infty} ar^n = a + a + a + \ldots + a + \ldots = \infty$

Consequently, the series diverges.

If $r = -1$, then $\sum_{n=0}^{\infty} ar^n = a - a + a - \ldots + a - \ldots$

In this case, observe that if N is even, then $S_N = a$, but if N is odd, then, $S_N = 0$.

Thus, the sequence of partial sums diverges (because $\lim_{n\to\infty} S_N \neq 0$) and the series itself must diverge. ■

Definition 3 *Series of the form:*

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \ldots + (-1)^n a_n + \ldots$$

or

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = +a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n + \ldots$$

are called Alternating Series.

Thm: (The **Alternating Series Test**) If $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ (or $\sum_{n=1}^{\infty} (-1)^n a_n$) is an alternating series with the properties that:

i) $|a_n| > |a_{n+1}|\ \forall n \in \mathbf{N}$, and

ii) $\lim_{n\to\infty} a_n = 0$,

then the series converges.

Proof. We will consider only the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. (The proof for $\sum_{n=1}^{\infty} (-1)^n a_n$ is similar.)

Let the hypotheses be given.

Consider the partial sum: $S_{2N+1} = a_1 \underbrace{-a_2 + a_3}_{<0} \underbrace{-a_4 + a_5}_{<0} - \ldots \underbrace{-a_{2N} + a_{2N+1}}_{<0} \le a_1$

i.e., $\forall N \in \mathbf{N}$, the partial sum S_{2N+1} is bounded above by a_1.

Now Consider the partial sum:

$$S_{2N} = \underbrace{a_1 - a_2}_{>0} + \underbrace{a_3 - a_4}_{>0} + \ldots + \underbrace{a_{2N-1} - a_{2N}}_{>0}$$

In particular, note that

$$S_{2N} < S_{2N} + \underbrace{a_{2N+1} - a_{2N+2}}_{>0} = S_{2N+2}$$

i.e., The subsequence of partial sums $\{S_2, S_4, S_6, \ldots, S_{2N}, \ldots\}$ is strictly monotone increasing.

Furthermore,

$$S_{2N} < S_{2N} + \underbrace{a_{2N+1}}_{>0} = S_{2N+1} \le a_1$$

i.e., The subsequence of partial sums $\{S_2, S_4, S_6, \ldots, S_{2N}, \ldots\}$ is bounded above by a_1.

Since the subsequence of partial sums $\{S_2, S_4, S_6, \ldots, S_{2N}, \ldots\}$ is strictly monotone increasing and bounded above, it converges to some limit, let's say S.

i.e., $\lim_{N\to\infty} S_{2N} = S$

Finally, we claim that the subsequence of partial sums $\{S_1, S_3, \ldots, S_{2N+1}, \ldots\}$ converges to S also. This can be shown as follows:

$$\lim_{N\to\infty} S_{2N+1} = \lim_{N\to\infty} (S_{2N} + a_{2N+1}) = \underbrace{\lim_{N\to\infty} S_{2N}}_{=S} + \underbrace{\lim_{N\to\infty} a_{2N+1}}_{=0} = S$$

Thus, the entire sequence of partial sums $\{S_1, S_2, S_3, \ldots, S_N, \ldots\}$ converges to S.

Hence, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. ■

EX The infinite series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1} \frac{1}{n} + \ldots$ is called the Alternating Harmonic Series.

Notice that $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$ (i.e., $a_n > a_{n+1} \forall n \in \mathbf{N}$)

Furthermore, notice that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ (i.e., $\lim_{n\to\infty} a_n = 0$)

Hence, the Alternating Harmonic Series converges.

Thm - The Direct Comparison Test - Part #1

Suppose that:

i) $0 \leq a_n < b_n$ $\forall n \in \mathbb{N}$, and

ii) $\sum_{n=1}^{\infty} b_n$ converges.

Then $\sum_{n=1}^{\infty} a_n$ converges also

Thm - The Direct Comparison Test - Part #2

Suppose that:

i) $0 \leq a_n < b_n$ $\forall n \in \mathbb{N}$, and

ii) $\sum_{n=1}^{\infty} a_n$ diverges.

Then $\sum_{n=1}^{\infty} b_n$ diverges also

Thm - The Limit Comparison Test

Given series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, suppose that: $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$.

Then either:

i) both series converge, or

ii) both series diverge.

Thm - The Integral Test: Given the series $\sum_{n=1}^{\infty} a_n$, suppose that a *continuous* function $f(x)$ has the properties that:

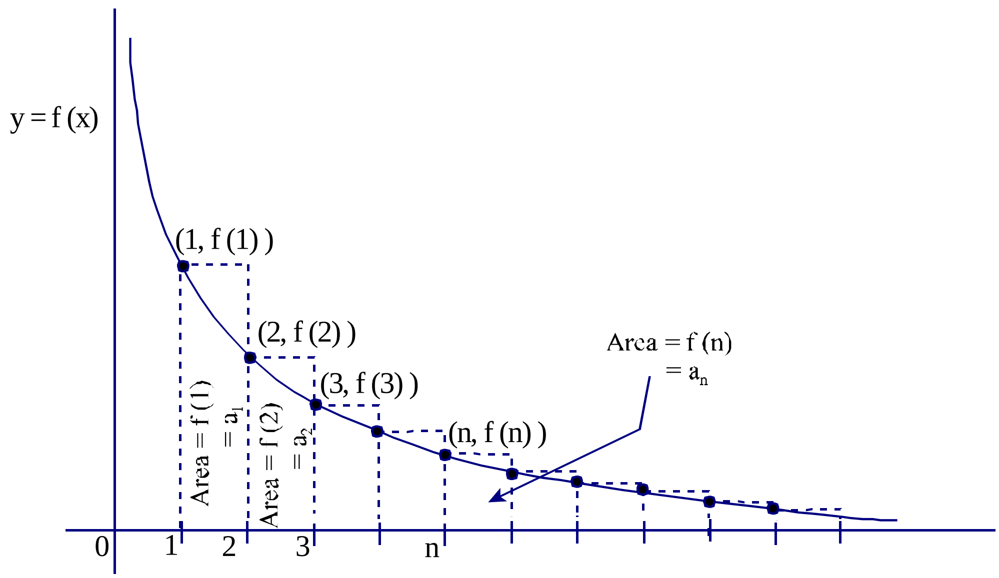
i) $f(x)$ is monotone decreasing on the interval $[1, \infty)$, and

i) $f(n) = a_n \forall n \in \mathbf{N}$

Then $\sum_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)\,dx$ converges and $\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^{\infty} f(x)\,dx$ diverges.

Proof. First, let's assume that $\int_1^{\infty} f(x)\,dx$ diverges.

Consider the picture below, and observe that, since the width of each rectangle equals one, the area of each rectangle is numerically equal to its height.



Observe that:

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\text{area of n}^{\text{th}} \text{ rectangle}\right)$

$\geq$ (area bounded by graph of $f(x)$ and the x-axis on the interval $[1, \infty)$)

$= \int_1^{\infty} f(x)\,dx$

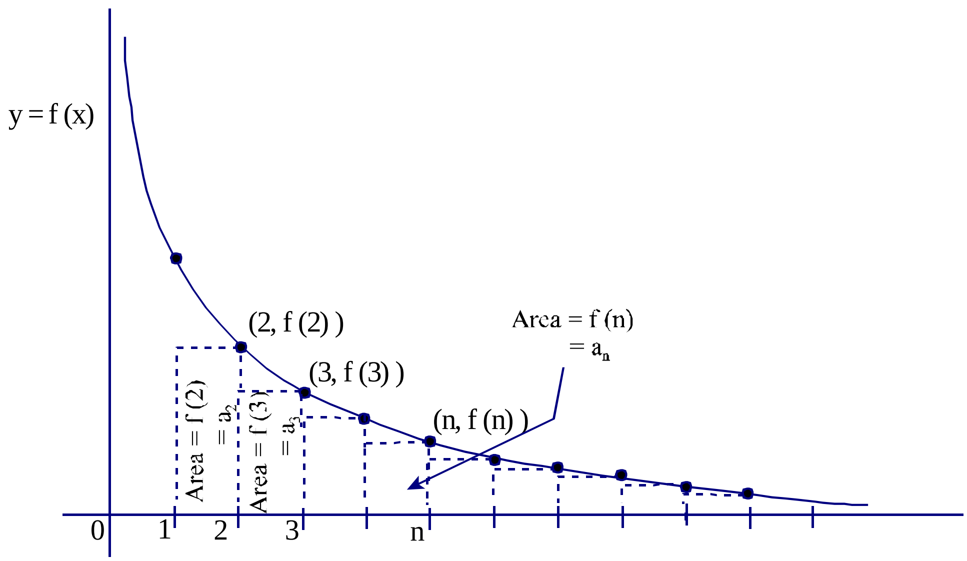
i.e., $\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\,dx = \infty$

i.e., If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also.

Next, let's assume that $\int_1^\infty f(x)\,dx$ converges.

Let's say that $\int_1^\infty f(x)\,dx = L$, for some $L \in \mathbf{R}$.

Consider the picture below, and again observe that, since the width of each rectangle equals one, the area of each rectangle is numerically equal to its height.



Observe that:

$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \left(\text{area of n}^{\text{th}} \text{ rectangle}\right)$

$\leq$ (area bounded by graph of $f(x)$ and the x-axis on the interval $[1,\infty)$)

$= \int_1^\infty f(x)\,dx$

i.e., $\sum_{n=2}^{\infty} a_n \leq \int_1^\infty f(x)\,dx = L$

Since all terms of the series are positive, the sequence of partial sums $\{S_1, S_2, S_3, \ldots, S_N, \ldots\}$ is monotone increasing.

Furthermore, the sequence of partial sums is bounded above by L.

Hence, the sequence of partial sums converges and, in turn, the series $\sum_{n=1}^{\infty} a_n$ converges.

i.e., If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also. ■

Def - The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p > 0$, is called the p**-series.**

Thm - (The p**-series Test)** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

Def - A series $\sum a_n$ **converges absolutely (or is absolutely convergent)** if $\sum |a_n|$ converges.

If the series $\sum a_n$ converges, but the series $\sum |a_n|$ diverges, then the series $\sum a_n$ is said to **converge conditionally (or is conditionally convergent).**

Ex - $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is conditionally convergent, since $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Ex - $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ is absolutely convergent, since $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-series test, with $p > 1$.

Thm - (Absolute Convergence Test) If an alternating series converges absolutely, then it converges (period)

Ex - $\sum (-1)^{n+1} \frac{1}{n^2}$ converges, becuse it converges absolutely (See the previous example)

Thm - The Ratio Test Suppose that $\sum a_n$ is a series in which all terms are either positive, or all terms are negative. Then:

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges

ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, the series diverges

iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Ex - $\sum \frac{2^n}{n!}$ converges by the Ratio Test

Observe:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)} = \lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)!}\right)\left(\frac{n!}{2^n}\right)\lim_{n\to\infty}\frac{2}{n+1} = 0$$

Hence, $\sum \frac{2^n}{n!}$ converges, because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

Ex - $\sum \frac{(2n)!}{100^n n!}$ diverges by the Ratio Test

Observe:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left(\frac{2(n+1)!}{100^{n+1}(n+1)!}\right)}{\left(\frac{(2n)!}{100^n n!}\right)} = \lim_{n\to\infty}\frac{(2n+2)!}{100^{n+1}(n+1)!}\frac{100^n n!}{(2n)!} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{100(n+1)}\frac{1}{1}$$

$$= \lim_{n\to\infty}(2n+2)\underbrace{\frac{(2n+1)}{100\,(n+1)}}_{=\frac{1}{50}} = \infty$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, the series $\sum \frac{(2n)!}{100^n n!}$ diverges by the Ratio Test.

Thm - n^{th} **Root Test** Suppose that $\sum_{n=1}^{\infty} a_n$ is a series in which all terms are positive. Then:

i) If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, the series converges

ii) If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$, the series diverges

iii) If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, the n^{th} Root Test is inconclusive

Remark: As an aid to applying the n^{th} **Root Test,** we have the following theorem:

Thm - $\lim_{n\to\infty} \sqrt[n]{n^k} = 1$ for any positive constant k,

Ex - $\sum_{n=1}^{\infty} \frac{3^n}{n^5}$ diverges by the n^{th} **Root Test**

Observe:

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^5}} = \lim_{n\to\infty} \left(\frac{3^n}{n^5}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(3^n)^{\frac{1}{n}}}{(n^5)^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{3}{(n)^{\frac{5}{n}}} = \frac{3}{1} = 3$

Since $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$, the series $\sum_{n=1}^{\infty} \frac{3^n}{n^5}$ (Diverges)

The Final Exam

1. Recall the Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Provide an example to show that the converse of the theorem does not hold.

(i.e., show that $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ **converges.)**

2. Recall: **Thm:** (The **Alternating Series Test)** If $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ (or $\sum_{n=1}^{\infty} (-1)^{n} a_n$) is an alternating series with the properties that:

i) $|a_n| > |a_{n+1}|$ $\forall n \in \mathbf{N}$, and

ii) $\lim_{n\to\infty} a_n = 0$,

then the series converges.

Provide/construct an example to show that the second property $(\lim_{n\to\infty} a_n = 0)$ **is necessary to guarantee convergence.**

3. Recall: **Thm -** The Geometric Series $\sum_{n=0}^{\infty} ar^n$

   i) Converges and has sum equal to $\frac{a}{1-r}$ if $|r| < 1$,

   ii) Diverges if $|r| \geq 1$

Show that given the geometric series $\sum_{n=k}^{\infty} ar^n = ar^k + ar^{k+1} + ar^{k+2} + \ldots$, **for some natural number** $k > 0$ **and** $|r| < 1$, **the series converges to** $\frac{ar^k}{1-r}$.

(i.e., $\sum_{n=k}^{\infty} ar^n$ **converges to** $\frac{1^{\text{st}} \text{ term}}{1-r}$, **where the "first term" is** ar^k.)

4. **Prove: The Direct Comparison Test - Part #1** (Hint: consider the sequence of partial sums and apply the Monotone (Bounded) Convergence Theorem.)

5. **Prove: The Direct Comparison Test - Part #2** (Hint: consider the sequence of partial sums and apply the Monotone (Bounded) Convergence Theorem.)

6. **Prove: The** p**-series Test**