MTH 1125 - Test #2 - Solutions

Summer 2015

Pat Rossi Name ________________________________

Instructions:

Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[\tan\left(x^3+3x\right)\right] =$

$$\begin{array}{|rcl|} \text{outer:} & = & \tan(\quad) \\ \text{deriv. of outer:} & = & \sec^2(\quad) \end{array}$$

$$\frac{d}{dx}\left[\underbrace{\tan}_{\text{outer}}\underbrace{\left(x^3+3x\right)}_{\text{inner}}\right] = \underbrace{\sec^2\left(x^3+3x\right)}_{\substack{\text{deriv. of outer}\\ \text{eval. at inner}}} \cdot \underbrace{\left(3x^2+3\right)}_{\substack{\text{deriv. of}\\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\tan\left(x^3+3x\right)\right] = \sec^2\left(x^3+3x\right)\cdot\left(3x^2+3\right)$

2. Suppose that $x=\sin(t)$ and that $t=8y^4$. Compute $\frac{dx}{dy}$ **using the Liebniz form** of the Chain Rule.

We Know:

$$\begin{array}{rcl} \frac{dx}{dt} & = & \cos(t) \\ \frac{dt}{dy} & = & 32y^3 \end{array}$$

We want: $\frac{dx}{dy}$

By the Liebniz form of the Chain Rule:

$$\frac{dx}{dy} = \frac{dx}{dt}\cdot\frac{dt}{dy} = \cos(t)\cdot\left(32y^3\right) = \cos\left(8y^4\right)\cdot\left(32y^3\right)$$

i.e., $\frac{dx}{dy} = \cos\left(8y^4\right)\cdot\left(32y^3\right)$

3. Compute: $\frac{d}{dx}\left[\left(4x^6+6x^4+24x\right)^{15}\right] =$

$$\frac{d}{dx}\left[\left(4x^6+6x^4+24x\right)^{15}\right] = \frac{d}{dx}\left[\underbrace{\left(4x^6+6x^4+24x\right)^{15}}_{(g(x))^n}\right] = \underbrace{15\left(4x^6+6x^4+24x\right)^{14}}_{\substack{\text{power rule}\\ \text{as usual}}}\cdot\underbrace{\left(24x^5+24x^3+24\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

i.e., $\frac{d}{dx}\left[\left(4x^6+6x^4+24x\right)^{15}\right] = 15\left(4x^6+6x^4+24x\right)^{14}\cdot\left(24x^5+24x^3+24\right)$

4. Compute: $\int \left(6x^3 - 2x^2 + 5x + 6 + 3\sqrt{x}\right) dx =$

$\int \left(6x^3 - 2x^2 + 5x + 6 + 3x^{\frac{1}{2}}\right) dx = 6\left[\frac{x^4}{4}\right] - 2\left[\frac{x^3}{3}\right] + 5\left[\frac{x^2}{2}\right] + 6x + 3\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] + C$

$\frac{3}{2}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 + 6x + 2x^{\frac{3}{2}} + C$

i.e., $\int \left(6x^3 - 2x^2 + 5x + 6 + 3\sqrt{x}\right) dx = \frac{3}{2}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 + 6x + 2x^{\frac{3}{2}} + C$

5. Given that $x^4 + 3x^3y^3 = 2y^2$; Compute y'

i) Differentiate both sides with respect to x

$\Rightarrow \frac{d}{dx}\left[x^4 + 3x^3y^3\right] = \frac{d}{dx}\left[2y^2\right]$

$\Rightarrow 4x^3 + \underbrace{9x^2 \cdot y^3 + 3y^2 \cdot y' \cdot 3x^3}_{\text{product rule}} = \underbrace{4y}_{\text{Power Rule as usual}} \cdot \underbrace{y'}_{\text{deriv of inner}}$

i.e., $4x^3 + 9x^2y^3 + 3y^2y'3x^3 = 4yy'$

ii) Solve for y' algebraically.

a) Get the y' terms on the left side, all other term on the right side

$\Rightarrow 3y^2y'3x^3 - 4yy' = -4x^3 - 9x^2y^3$

b) Factor out y'

$\Rightarrow \left(3y^2 3x^3 - 4y\right) y' = -4x^3 - 9x^2y^3$

c) Divide by the "cofactor" of y'

$\Rightarrow y' = \frac{-4x^3 - 9x^2y^3}{3y^2 3x^3 - 4y}$

i.e., $y' = \frac{-4x^3 - 9x^2y^3}{3y^2 3x^3 - 4y} = -\frac{4x^3 + 9x^2y^3}{9y^2x^3 - 4y}$

6. $f(x) = x^3 + 3x^2 - 9x + 4$

i) Determine the intervals on which $f(x)$ is increasing/decreasing

ii) Identify all relative maximums and minimums

i) Compute $f'(x)$ and find the critical numbers.

$f'(x) = 3x^2 + 6x - 9$

a) "Type a" $(f'(c) = 0)$

$\Rightarrow f'(x) = 3x^2 + 6x - 9 = 0$

$\Rightarrow 3x^2 + 6x - 9 = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0$

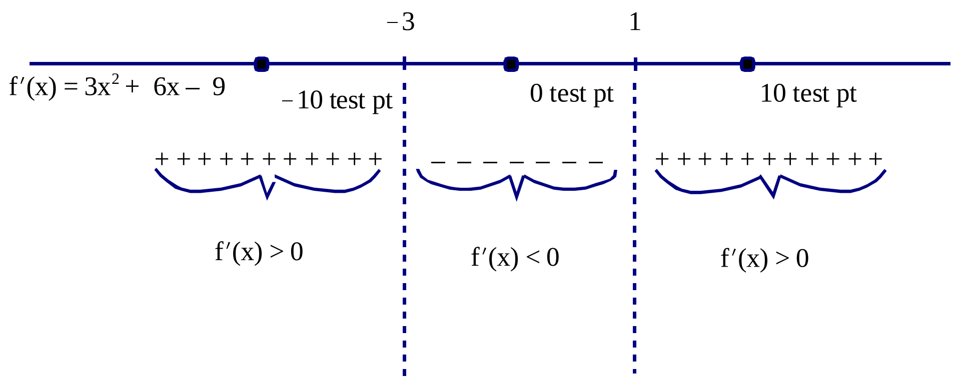
$\Rightarrow x = -3,\ x = 1$ are "type a" critical numbers

b) "Type b" ($f'(c)$ undefined)

There are none.

ii) Draw a "sign graph" of $f'(x)$

iii) From each interval. select a "sample point" and plug into $f'(x)$.



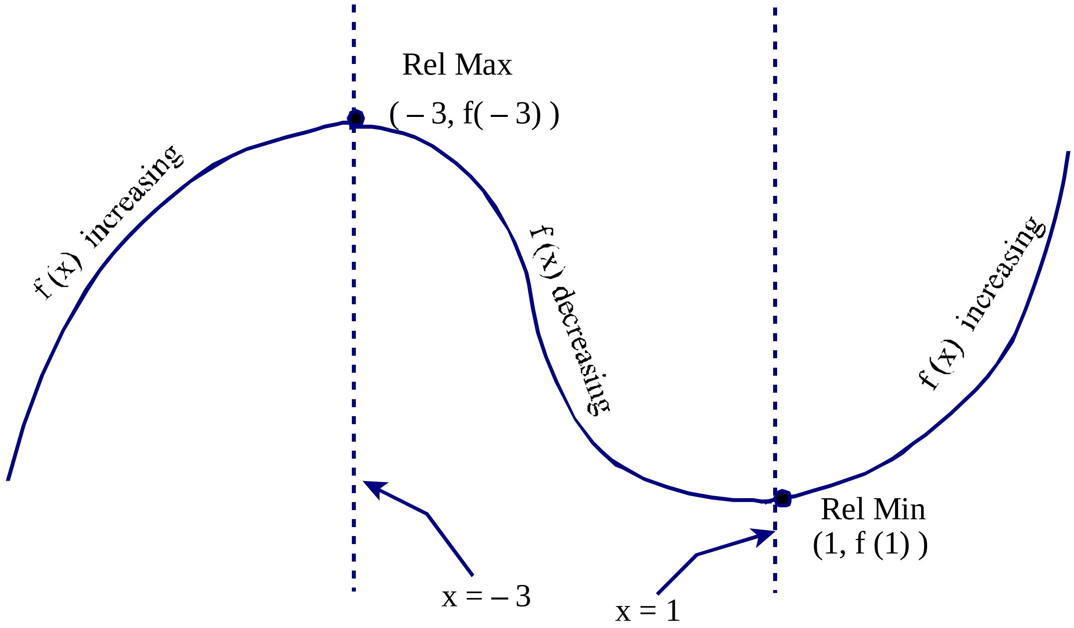
$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(1, \infty)$

(because $f'(x) > 0$ on these intervals).

$f(x)$ is **decreasing** on the interval $(-3, 1)$

(because $f'(x) < 0$ on this interval).

iv) Sketch a rough graph of $f(x)$ to find the relative maxes and mins.



From the graph of $f(x) = x^3 - 3x^2 - 9x + 4$ it is clear that:

$(-3, f(-3)) = (-3, 31)$ is a **relative max.**, and

1. $(1, f(1)) = (1, -1)$ is a **relative min**

7. $f(x) = 3x^{\frac{2}{3}} - 2x$

i) Determine the intervals on which $f(x)$ is increasing/decreasing

ii) Identify all relative maximums and minimums

i) Compute $f'(x)$ and find the critical numbers.

$f'(x) = 2x^{-\frac{1}{3}} - 2 = \frac{2}{x^{\frac{1}{3}}} - 2 = \frac{2}{x^{\frac{1}{3}}} - \frac{2}{1} \cdot \frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{2-2x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

i.e., $f'(x) = \frac{2-2x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

a) "Type a" ($f'(c) = 0$)

$\Rightarrow f'(x) = \frac{2-2x^{\frac{1}{3}}}{x^{\frac{1}{3}}} = 0$

$\Rightarrow 2 - 2x^{\frac{1}{3}} = 0$

$\Rightarrow 2x^{\frac{1}{3}} = 2$

$\Rightarrow x^{\frac{1}{3}} = 1$

$\Rightarrow x = 1$ is a "type a" critical number

b) "Type b" ($f'(c)$ undefined)

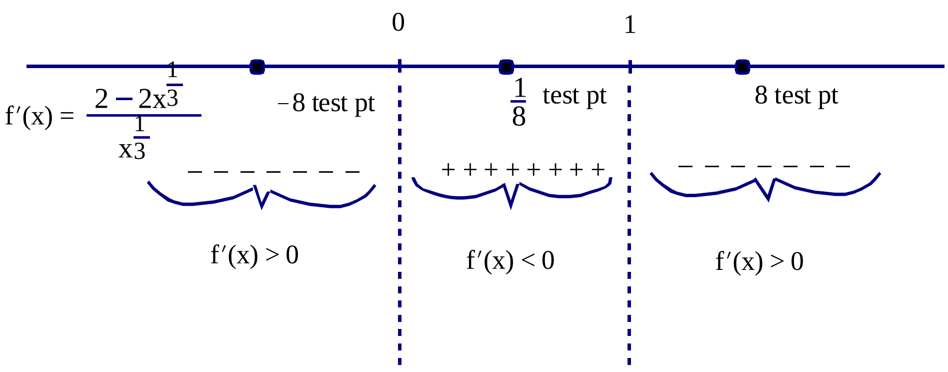
Set $x^{\frac{1}{3}} = 0$

$\Rightarrow x = 0$ is a "type b" critical number

ii) Draw a "sign graph" of $f'(x)$

iii) From each interval. select a "sample point" and plug into $f'(x)$.

(a)



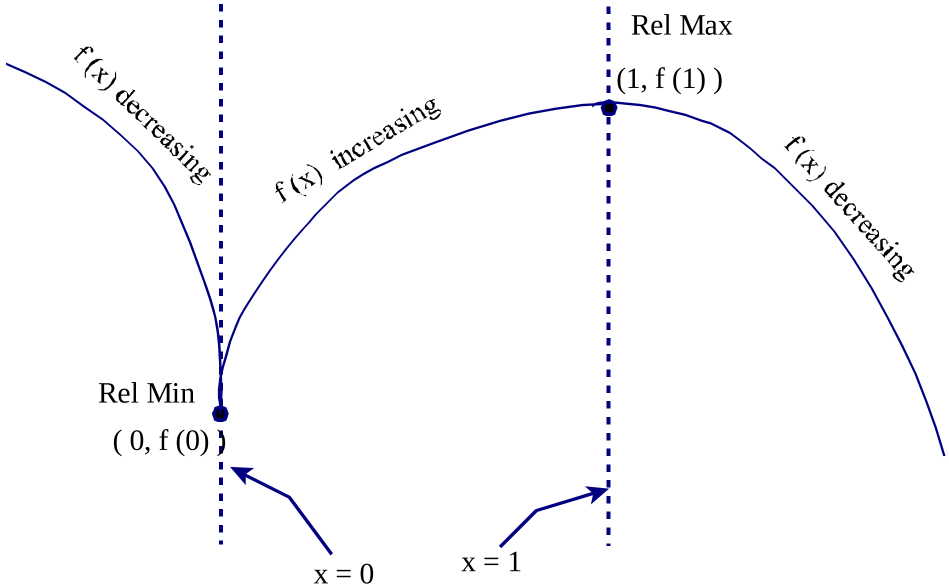
$f(x)$ is **increasing** on the interval $(0, 1)$

(because $f'(x) > 0$ on this interval).

$f(x)$ is **decreasing** on the intervals $(-\infty, 0)$ and $(1, \infty)$

(because $f'(x) < 0$ on these intervals).

iv) Sketch a rough graph of $f(x)$ to find the relative maxes and mins.



From the graph of $f(x) = 3x^{\frac{2}{3}} - 2x$ it is clear that:

$(0, f(0)) = (0, 0)$ is a **relative min.**, and

$(1, f(1)) = (1, 1)$ is a **relative max**

8. $f(x) = x^3 + 3x^2 - 24x + 6$ on the interval $[-3, 3]$. Find the absolute maximum and absolute minimum values.

Note: $f(x)$ is [1]continuous (no zero divides) on the [2]closed, [3]finite interval $[-3, 3]$.

Hence, we can use the Absolute Max/Min Value Test.

1. Find Critical numbers

   $f'(x) = 3x^2 + 6x - 24$

   a) (Type a)

      $f'(x) = 3x^2 + 6x - 24 = 0$

      $\Rightarrow x^2 + 2x - 8 = 0$

      $\Rightarrow (x + 4)(x - 2) = 0$

      $\Rightarrow x = -4$ and $x = 2$ are "type a" critical numbers

      Since $x = -4$ is NOT in the interval $[-3, 3]$, we discard it as a critical number

   b) (Type b)

      None

ii) Plug Critical numbers and endpoints into the *original function.*

   $f(-3) = (-3)^3 + 3(-3)^2 - 24(-3) + 6 = 78$ $\Leftarrow$ Abs. Max. Value

   $f(2) = (2)^3 + 3(2)^2 - 24(2) + 6 = -22$ $\Leftarrow$ Abs. Min. Value

   $f(3) = (3)^3 + 3(3)^2 - 24(3) + 6 = -12$

   The absolute maximum value is 78 (attained at $x = -3$)

   The absolute minimum value is -22 (attained at $x = 2$)

9. Compute: $\int \left(8x^3+6x^2\right)^{10}\left(2x^2+x\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(8x^3+6x^2\right)^{10}$ ↗ inner, ↑ outer

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(8x^3+6x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(8x^3+6x^2\right)}_{\text{function}} - - - - - \rightarrow \underbrace{\left(2x^2+x\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(8x^3+6x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 8x^3+6x^2 \\ \Rightarrow \tfrac{du}{dx} &= 24x^2+12x \\ \Rightarrow du &= \left(24x^2+12x\right)dx \\ \Rightarrow \tfrac{1}{12}du &= \left(2x^2+x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(8x^3+6x^2\right)^{10}}_{u^{10}}\underbrace{\left(2x^2+x\right)dx}_{\frac{1}{12}du} = \int u^{10}\,\tfrac{1}{12}du = \tfrac{1}{12}\int u^{10}du$$

4. Integrate (in terms of u).

$$\tfrac{1}{12}\int u^{10}du = \tfrac{1}{12}\left[\tfrac{u^{11}}{11}\right]+C = \tfrac{1}{132}u^{11}+C$$

5. Re-express in terms of the original variable, x.

$$\int \left(8x^3+6x^2\right)^{10}\left(2x^2+x\right)dx = \underbrace{\frac{1}{132}\left(8x^3+6x^2\right)^{11}+C}_{\frac{1}{132}u^{11}+C}$$

i.e., $\int \left(8x^3+6x^2\right)^{10}\left(2x^2+x\right)dx = \frac{1}{132}\left(8x^3+6x^2\right)^{11}+C$

10. $f(x) = x^4 + 2x^3 - 36x^2 + 12x + 12$

i) Determine the intervals on which $f(x)$ is concave up/concave down

ii) Identify all points of inflection

1. Compute $f''(x)$ and find the possible points of inflections.

   $f'(x) = 4x^3 + 6x^2 - 72x + 12$

   $f''(x) = 12x^2 + 12x - 72$

   a) "Type a" $(f''(c) = 0)$

   $\Rightarrow f''(x) = 12x^2 + 12x - 72 = 0$

   $\Rightarrow x^2 + x - 6 = 0$

   $\Rightarrow (x+3)(x-2) = 0$

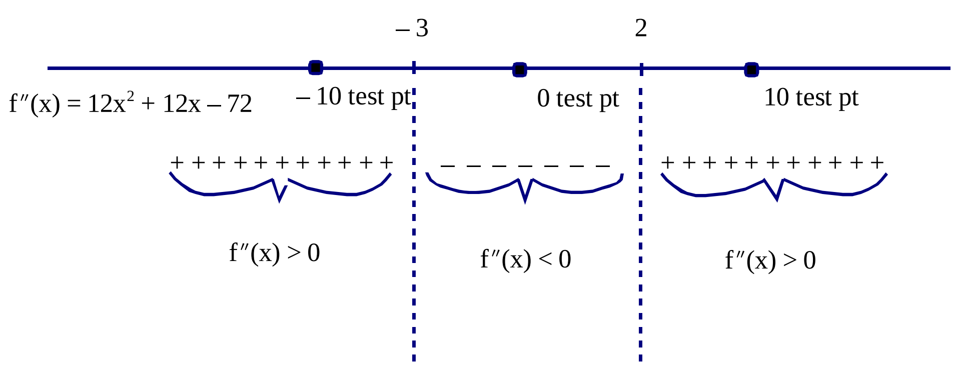
   $\Rightarrow x = -3,\ x = 2$ are possible "type a" points of inflection

   b) "Type b" $\left(f''(c) \text{ undefined}\right)$

   There are none.

2. Draw a "sign graph" of $f''(x)$

3. From each interval, select a "test point" and plug into $f''(x)$.



$f(x)$ is **concave up** on the intervals $(-\infty, -3)$ and $(2, \infty)$

(because $f''(x) > 0$ on these intervals).

$f(x)$ is **concave down** on the interval $(-3, 2)$

(because $f''(x) < 0$ on this interval).

Since $f(x)$ changes concavity at $x = -3$ and $x = 2$, the points

$(-3, f(-3)) = (-3, -321)$ and $(2, f(2)) = (2, -76)$ are points of inflection.

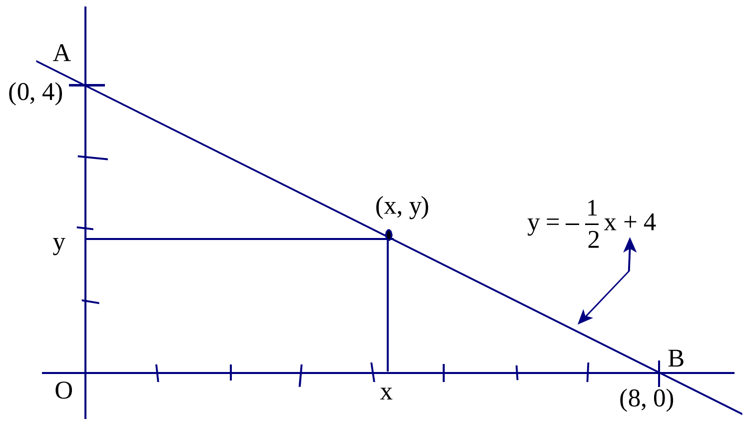
11. A rectangle is inscribed within triangle AOB, such that:

i) The base of the rectangle rests on the positive x-axis

ii) One side of the rectangle borders the positive y-axis

iii) The vertex of the rectangle that is opposite the origin lies on the graph of $y = -\frac{1}{2}x + 4$

What should the value of x be so that the inscribed rectangle has the largest area possible?



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a) Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $y = -\frac{1}{2}x + 4$.

   Hence, the y-coordinate of the point (x, y) is $y = -\frac{1}{2}x + 4$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x\left(-\frac{1}{2}x + 4\right) = -\frac{1}{2}x^2 + 4x$

   i.e., $A(x) = -\frac{1}{2}x^2 + 4x$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 8$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 8]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = -x + 4$

      a. "Type a" $(A'(c) = 0)$

         $\Rightarrow A'(x) = -x + 4 = 0$

         $\Rightarrow x = 4$ is a critical number

      b. "Type b" ($A'(c)$ is undefined)

         Look for x-values that cause division by zero in $A'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = -\frac{1}{2}(0)^2 + 4(0) = 0$

      $A(4) = -\frac{1}{2}(4)^2 + 4(4) = 8 \longleftarrow$ Abs Max Value

      $A(8) = -\frac{1}{2}(8)^2 + 4(8)$

5. Make sure that we've answered the original question.

   "Determine the value of x ..."

   $x = 4$