# MTH 1125 (12 pm) Test #3 Pod A - Solutions

FALL 2020

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**Instructions.**   Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 - 3x^2 + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. (Caution - there are **two** critical numbers. Make sure you get them both!)

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 6x^2 - 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for $x$

   $\Rightarrow f'(x) = 6x^2 - 6x = 0$

   $\Rightarrow 6x(x - 1) = 0$

   $\Rightarrow 6x = 0$   or   $(x - 1) = 0$

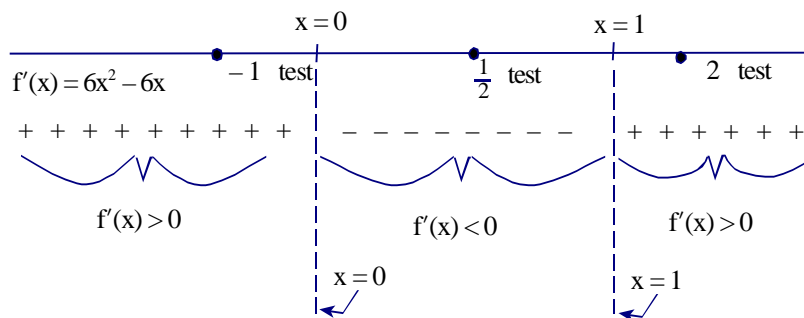   $\Rightarrow x = 0$ and $x = 1$ are critical numbers.

   b. "Type b" $(f'(c)$ is undefined)

   Look for $x$-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the $x$-axis
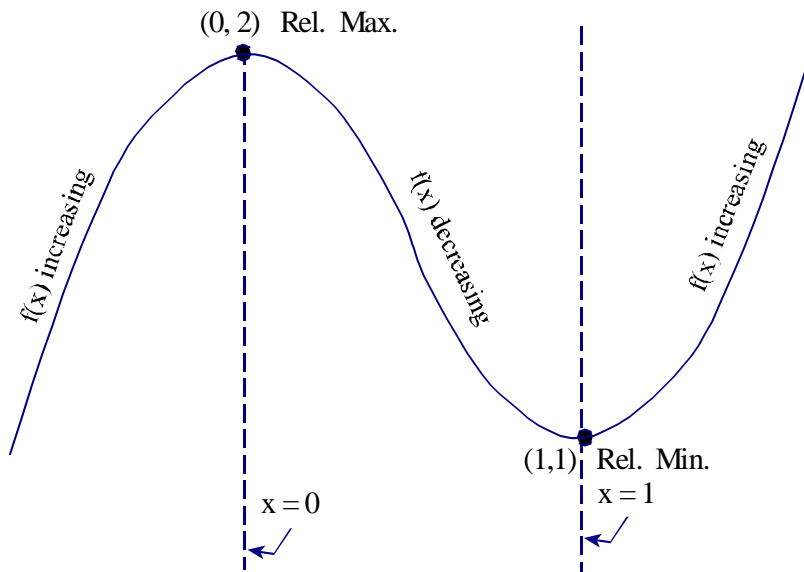
3. Pick a "test point" from each interval to plug into $f'(x)$

$f(x)$ is **increasing** on the interval(s) $(-\infty, 0)$ and $(1, \infty)$
(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(0, 1)$
(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

$(0, 2)$ Rel. Max.

f(x) increasing

f(x) decreasing

f(x) increasing

(1,1) Rel. Min.

x = 0

x = 1

**Rel Max** $(0, f(0)) = (0, 2)$

**Rel Min** $(1, f(1)) = (1, 1)$

2. $f(x) = \frac{1}{4}x^4 - 2x^3 - \frac{15}{2}x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

1. Compute $f''(x)$ and find possible points of inflection.

   $f'(x) = x^3 - 6x^2 - 15x + 6$

   $f''(x) = 3x^2 - 12x - 15$

   Find possible points of inflection:

   a. "Type a" $(f''(x) = 0)$

      Set $f''(x) = 0$

      $\Rightarrow f''(x) = 3x^2 - 12x - 15 = 0$

      $\Rightarrow 3(x^2 - 4x - 5) = 0$

      $\Rightarrow 3(x + 1)(x - 5) = 0$
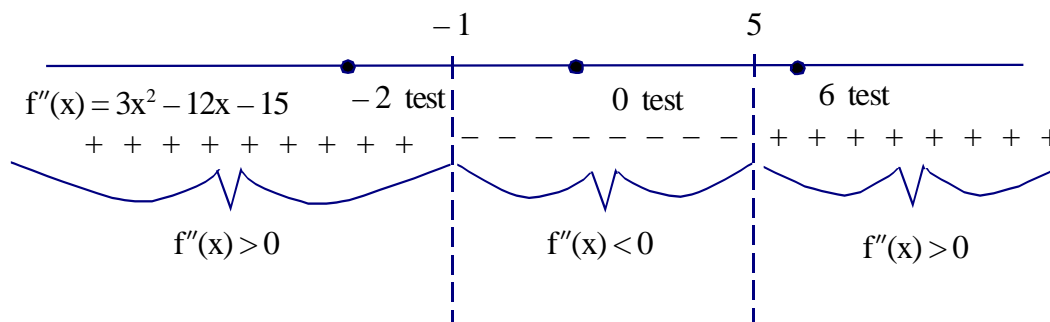
      $x = -1, 5$ possible "type a" points of inflection

   b. "Type b" $(f''(x)$ undefined$)$

      No "Type b" points of inflection

2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the $x$-axis.

3. Select a test point from each interval and plug into $f''(x)$

$$
\begin{array}{ccc}
 & -1 & 5 \\
\end{array}
$$

f''(x) = 3x² – 12x – 15    –2 test    0 test    6 test

+ + + + + + + + +     – – – – – – – –     + + + + + + + + +

f''(x) > 0          f''(x) < 0          f''(x) > 0

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$f(x)$ is **concave up** on the intervals $(-\infty, -1)$ and $(5, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-1, 5)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -1$ and $x = 5$, the points:
$(-1, f(-1)) = \left(-1, -\frac{33}{4}\right)$
          and
$(5, f(5)) = \left(5, -\frac{993}{4}\right)$ **are** points of inflection.

3. $f(x) = x^3 - 6x^2 - 15x + 3$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

   Note: $^1f(x)$ is continuous (since it is a polynomial) on the $^2$closed, $^3$finite interval $[-2, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

   i. Compute $f'(x)$ and find the critical numbers.

   $f'(x) = 3x^2 - 12x - 15$

   a. "Type a" $(f'(x) = 0)$

   $f'(x) = 3x^2 - 12x - 15 = 0$

   $\Rightarrow 3(x^2 - 4x - 5) = 0$

   $\Rightarrow (x+1)(x-5) = 0$

   $\Rightarrow x = -1, 5$ are "type a" critical numbers

   Since $x = 5$ is not in the interval $[-2, 2]$, we discard it as a critical number.

   b. "Type b" $(f'(x)$ is undefined)

   No "Type b" critical numbers

   ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

   $f(-2) = (-2)^3 - 6(-2)^2 - 15(-2) + 3 = -2$

   $f(-1) = (-1)^3 - 6(-1)^2 - 15(-1) + 3 = 11$ $\leftarrow$ Abs Max Value

   $f(2) = (2)^3 - 6(2)^2 - 15(2) + 3 = -43$ $\leftarrow$ Abs Min Value

   > The Abs Max Value is 11
   > (attained at $x = -1$)
   >
   > The Abs Min Value is $-43$
   > (attained at $x = 2$)

4. $f(x) = \frac{3}{10}x^{\frac{20}{7}} - x^{\frac{6}{7}} - 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

$$f'(x) = \frac{6}{7}x^{\frac{13}{7}} - \frac{6}{7}x^{-\frac{1}{7}} = \frac{6x^{\frac{13}{7}}}{7} - \frac{6}{7x^{\frac{1}{7}}} = \frac{6x^{\frac{13}{7}}}{7}\frac{x^{\frac{1}{7}}}{x^{\frac{1}{7}}} - \frac{6}{7x^{\frac{1}{7}}} = \frac{6x^2-6}{7x^{\frac{1}{7}}}$$

i.e., $f'(x) = \frac{6x^2-6}{7x^{\frac{1}{7}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 0$ and solve for $x$

$\Rightarrow f'(x) = \frac{6x^2-6}{7x^{\frac{1}{7}}} = 0$

$\Rightarrow 6x^2 - 6 = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x+1)(x-1) = 0$

$\Rightarrow x = -1$ and $x = 1$ are critical numbers.
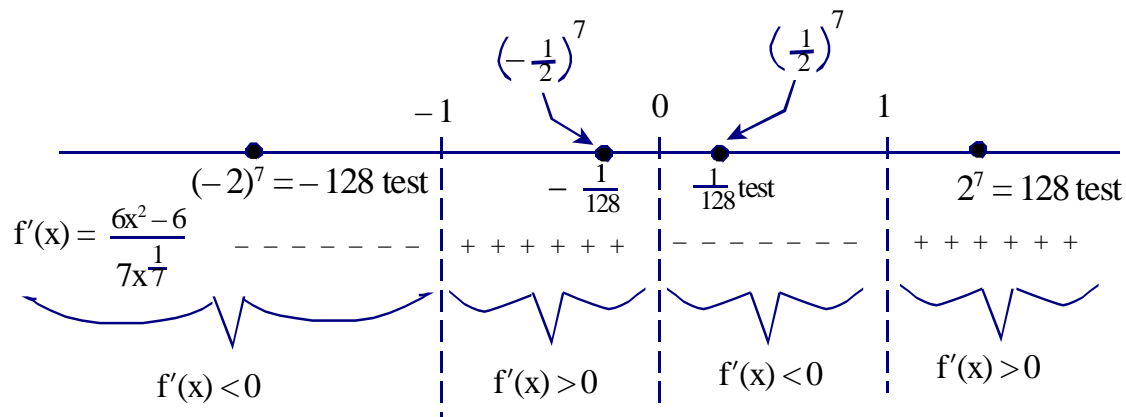
b. "Type b" $(f'(c)$ is undefined)

Look for $x$-value that causes division by zero.

$\Rightarrow 7x^{\frac{1}{7}} = 0$

$\Rightarrow x = 0$ "type b" critical number

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the $x$-axis

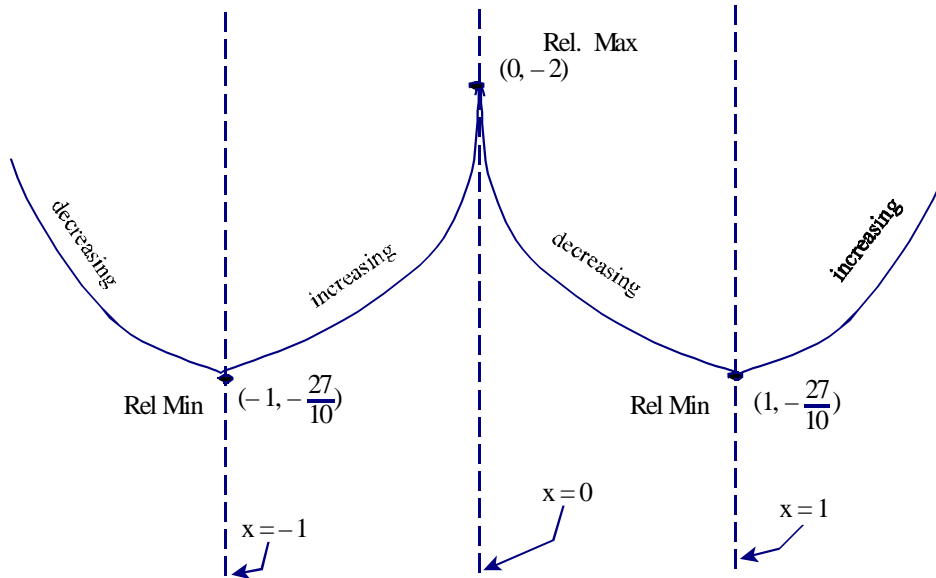3. Pick a "test point" from each interval to plug into $f'(x)$



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$f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$

(because $f'(x)$ is negative on those intervals)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel. Max $(0, -2)$

decreasing

increasing

decreasing

increasing

Rel Min $\left(-1, -\frac{27}{10}\right)$

Rel Min $\left(1, -\frac{27}{10}\right)$

$x = 0$

$x = 1$

$x = -1$

**Rel Minimums:** $(-1, f(-1)) = \left(-1, -\frac{27}{10}\right)$

and $(1, f(1)) = \left(1, -\frac{27}{10}\right)$

**Rel Maximum:** $(0, f(0)) = (0, -2)$

5. A rectangle is inscribed in the region bounded by the positive $x$-axis, the positive $y$-axis, and the graph of $f(x) = (x - 14)^6$ as shown below. Determine the value of $x$ that makes the area of the rectangle as large as possible.

**REMARK:** When you create the area function $A(x)$, **do not simplify (i.e. "multiply it out") before you compute the derivative.** If you simplify (i.e. "multiply it out") the area function $A(x)$ before you compute the derivative, it will be very difficult to find the critical numbers.
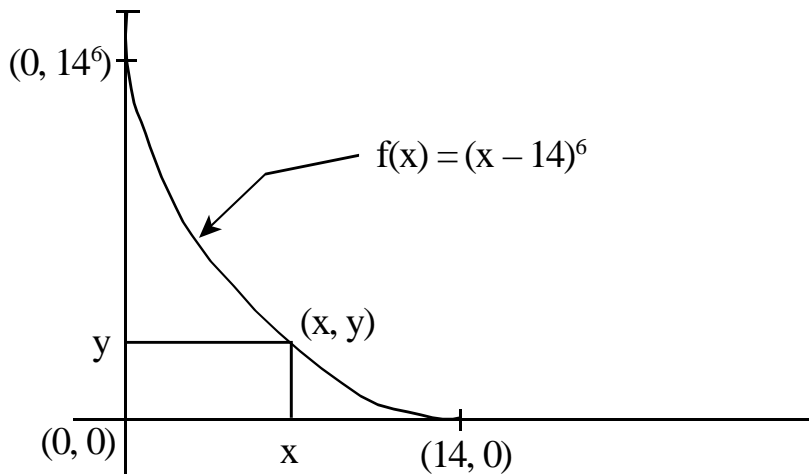


1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

   (Done)

2. Express $A$ as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point $(x, y)$ must be on the graph of $f(x) = (x - 14)^6$.

   Hence, the $y$-coordinate of the point $(x, y)$ is $y = (x - 14)^6$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(x - 14)^6$

3. Determine the restrictions on the independent variable $x$.

   From the picture, $0 \le x \le 14$

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4. Maximize $A(x)$, using the techniques of Calculus.

Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 14]$.

Thus, we can use the Absolute Max/Min Value Test.

1. Find the critical numbers.
$$A'(x) = \frac{d}{dx}\left[x(x-14)^6\right] = \underbrace{(x-14)^6 + 6x(x-14)^5}_{\text{Product Rule}}$$

i.e., $A'(x) = (x-14)^6 + 6x(x-14)^5$

a. "Type a" $(f'(c) = 0)$
$$\Rightarrow A'(x) = (x-14)^6 + 6x(x-14)^5 = 0$$
$$\Rightarrow (x-14)^6 + 6x(x-14)^5 = 0$$
$$\Rightarrow (x-14)^5\left[(x-14) + 6x\right] = 0$$
$$\Rightarrow (x-14)^5(7x-14) = 0$$
$$\Rightarrow x = 14 \text{ and } x = 2 \text{ are critical numbers}$$

b. "Type b" $(f'(c)$ is undefined)

Look for $x$-values that cause division by zero in $f'(x)$

(None)

2. Plug critical numbers and endpoints into the *original* function.
$$A(0) = (0)((0) - 14)^6 = 0$$
$$A(2) = (2)((2) - 14)^6 = 2 \cdot 12^6 = \text{a really big number} \longleftarrow \text{Abs Max Value}$$
$$A(14) = (14)((14) - 14)^4 = 0$$

5. Make sure that we've answered the original question.

"Determine the value of $x$ ..."

$x = 2$