

MTH 4441 Exercises To study for Test #1 - Solutions

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1. In each case below, determine whether $*$ is a **closed** binary operation on the given set. If it *IS* a **closed** binary operation, then determine whether it is commutative and/or associative.

(a) $(\mathbb{Z}, *)$ where $a * b = a + b^2$

IS a **closed** binary operation on \mathbb{Z}

NOT commutative. Given $a \neq b$, $a * b = a + b^2 \neq b + a^2 = b * a$

NOT associative.

$$(a * b) * c = (a + b^2) * c = (a + b^2) + c^2 = a + b^2 + c^2$$

$$a * (b * c) = a * (b + c^2) = a + (b + c^2)^2 = a + b^2 + 2bc^2 + c^4$$

$$(a * b) * c \neq a * (b * c)$$

(b) $(\mathbb{Z}, *)$ where $a * b = a^2b^3$

IS a **closed** binary operation on \mathbb{Z}

NOT commutative. Given $a \neq b$, $a * b = a^2b^3 \neq b^2a^3 = b * a$

NOT associative.

$$(a * b) * c = a^2b^3 * c = (a^2b^3)^2 c^3 = a^4b^6c^3$$

$$a * (b * c) = a * b^2c^3 = a^2 (b^2c^3)^3 = a^2b^6c^9$$

$$(a * b) * c \neq a * (b * c)$$

(c) $(\mathbb{R}, *)$ where $a * b = b$

IS a **closed** binary operation on \mathbb{R}

Is NOT Commutative: Given $a \neq b$, $a * b = b \neq a = b * a$

i.e., $a * b \neq b * a$

IS Associative:

$$(a * b) * c = b * c = c$$

$$a * (b * c) = a * c = c$$

i.e., $(a * b) * c = a * (b * c)$

(d) $(S, *)$ where $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ and $*$ is matrix addition

IS a **closed** binary operation on S

IS Commutative: Given $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, we have:

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} = \begin{bmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{bmatrix} = B + A$$

i.e., $A + B = B + A$

IS Associative: Given $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$; $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$, we have:

$$\begin{aligned} (A + B) + C &= \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 & (a_2 + b_2) + c_2 \\ (a_3 + b_3) + c_3 & (a_4 + b_4) + c_4 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + (b_1 + c_1) & a_2 + (b_2 + c_2) \\ a_3 + (b_3 + c_3) & a_4 + (b_4 + c_4) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 + c_1 & b_2 + c_2 \\ b_3 + c_3 & b_4 + c_4 \end{bmatrix} \\ &= A + (B + C) \end{aligned}$$

i.e., $(A + B) + C = A + (B + C)$

2. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, and let (\mathbb{Z}_6, \oplus) be a group, where \oplus is addition modulo 6. Construct the group table.

\oplus	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

3. In the group (\mathbb{Z}_6, \oplus) , what is the order of the element 2? What is the order of the element 3?

(i.e., $o(2) = ?$ $o(3) = ?$)

$$1 \cdot 2 = 2; 2 \cdot 2 = 4; 3 \cdot 2 = 0; \quad o(2) = 3$$

$$1 \cdot 3 = 3; 2 \cdot 3 = 0; \quad o(3) = 2$$

4. Construct the group table for (\mathbb{Z}_7, \oplus) .

\oplus	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

5. Let $U_5 = \{1, 2, 3, 4\}$, and let (U_5, \odot) be a group, where \odot is multiplication modulo 5. Construct the group table.

In (U_5, \odot) , the operation \odot is multiplication modulo 5

\odot	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

6. Construct the group table for (U_3, \odot) .

In (U_3, \odot) , the operation \odot is multiplication modulo 3

\odot	1	2
1	1	2
2	2	1

7. Construct the group table for (U_7, \odot) .

In (U_7, \odot) , the operation \odot is multiplication modulo 7

\odot	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

8. Construct the group table for (U_6, \odot) .

\odot	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

(a) (U_6, \odot) is NOT a group. Give at least two reasons why it is not a group

1. U_6 is not closed under \odot . For example: $2 \odot 3 = 0$
2. None of the elements of U_6 appear at least once in every row and in every column. For example, 1 does not appear in the row headed by 2
3. The elements 2, 3, 4 appear more than once in some rows and columns. For example, 3 appears three times in the row and the column headed by 3.
4. The elements 2, 3, 4 do not have an inverse. This can be seen by the fact that the identity 1 does not appear in the rows and the columns that are headed by 2, 3, and 4.

9. Determine whether the table below defines a group for $G = \{a, b, c\}$. (State why or why not.)

*	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

This table does NOT define $(G, *)$ as a group.

1. The elements b and c appear more than once in the rows and columns headed by b and c .
2. The identity 1 does not appear in any row or column headed by b or c , which indicates that neither b nor c has an inverse.

10. Determine whether the table below defines a group for $G = \{a, b, c, d, e, f\}$. State why or why not. (You may assume that the operation $*$ is associative.)

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	d	f	a	c	e
c	c	f	b	e	a	d
d	d	a	e	b	f	c
e	e	c	a	f	d	b
f	f	e	d	c	b	a

This table **does** define a group.

The element a is **clearly the identity**, as $a * x = x$, and $x * a = x$. $\forall x \in G$

No elements outside of the set G appear in the table. This means that $x * y \in G$, $\forall x, y \in G$. Hence, **the set G is closed under $*$** .

Finally, the identity appears in each row and column, indicating that **each element of G has an inverse**. Furthermore, each right inverse is also the left inverse. (i.e., if $x * y = a$, then $y * x = a$)

11. In the previous exercise, what is the inverse of d ? How do you know?

The inverse of d is b . The element a is the identity, and $b * d = a$ and $d * b = a$

12. Compute the remainder of 25 modulo 7 (i.e. $25 \equiv \underline{\quad} \pmod{7}$)

$$25 = (3)(7) + 4, \quad \text{Hence } 25 \equiv \underline{4} \pmod{7}$$

13. Compute the remainder of 48 modulo 5 (i.e. $48 \equiv \underline{\quad} \pmod{5}$)

$$48 = (9)(5) + 3, \quad \text{Hence } 48 \equiv \underline{3} \pmod{5}$$

14. Compute the remainder of 53 modulo 14 (i.e. $53 \equiv \underline{\quad} \pmod{14}$)

$$53 = (3)(14) + 11, \quad \text{Hence } 53 \equiv \underline{11} \pmod{14}$$

15. Determine whether 58 and 75 are congruent modulo 9 (Determine whether $58 \equiv 75 \pmod{9}$)

Method #1 $(58 - 75) = -17 \neq n(9), \forall n \in \mathbb{Z}$. Hence $58 \not\equiv 75 \pmod{9}$

Method #2 $58 = (6)(9) + 4$, and $75 = (8)(9) + 3$

Since 58 and 75 do not have the same remainder when divided by 9, $58 \not\equiv 75 \pmod{9}$

16. Determine whether 43 and 59 are congruent modulo 16 (Determine whether $43 \equiv 59 \pmod{16}$)

Method #1 $(43 - 59) = -16 = (1)(16)$, $43 \equiv 59 \pmod{16}$

Method #2 $43 = (2)(16) + 11$, and $59 = (3)(16) + 11$

Since 43 and 59 have the same remainder when divided by 16, $43 \equiv 59 \pmod{16}$

17. Compute $\gcd(4, 18)$

We will factor both into prime factors

$$4 = 2^2 \text{ and } 18 = 2 \cdot 3^2$$

Therefore, 2 is the greatest common divisor, or factor ($\gcd(4, 18) = 2$)

18. Compute $\gcd(25, 40)$

We will factor both into prime factors

$$25 = 5^2 \text{ and } 40 = 2^3 \cdot 5$$

Therefore, 5 is the greatest common divisor, or factor ($\gcd(25, 40) = 5$)

19. Compute $\gcd(4, 25)$

We will factor both into prime factors

$$4 = 2^2 \text{ and } 25 = 5^2$$

4 and 25 have no prime factors in common. Hence, $\gcd(4, 25) = 1$