MTH 3311 - Test #2

Spring 2021

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From the exercises below, do two.

1. It's really hot out. I've been clearing brush for hours and I'm really thirsty. I come inside to get a cold beer from the "fridge," and there's no beer in the "fridge" – I forgot to put the beer in there. I have some cans of beer on the kitchen counter, but they're at room temperature ($80\,^{\circ}\mathrm{F}$ – Yeah, I forgot to turn on the AC also!) I put a few cans of beer in the freezer, which is kept at a constant $-10\,^{\circ}\mathrm{F}$. At $T = 10$ minutes later, the temperature of each beer is $60\,^{\circ}\mathrm{F}$. How long (starting from the time that I put the beer in the freezer) will it take for my beer to be at a temperature of $28\,^{\circ}\mathrm{F}$? (Leave your answer in terms of natural logs.)

   By Newton's Law of Cooling, the rate at which the temperature of the beer changes is proportional to the difference of the temperature of the beer T and the temperature inside the freezer, T_f.

   $\Rightarrow \frac{dT}{dt} = k\left(T_f - T\right)$

   $\Rightarrow \frac{dT}{dt} + kT = kT_f$

   $\Rightarrow e^{\int k dt} = e^{kt}$

   $\Rightarrow e^{kt}\frac{dT}{dt} + ke^{kt}T = kT_f e^{kt}$

   $\Rightarrow \frac{d}{dt}\left[e^{kt}T\right] = kT_f e^{kt}$

   $\Rightarrow \int \frac{d}{dt}\left[e^{kt}T\right] dt = \int ke^{kt}T_f dt$

   $\Rightarrow e^{kt}T = k\left(\frac{1}{k}e^{kt}\right)T_f + C = e^{kt}T_f + C$

   i.e., $e^{kt}T = e^{kt}T_f + C$

   $\Rightarrow T = T_f + Ce^{-kt}$

   Recall that at $t = 0$ min, $T = 80\,^{\circ}\mathrm{F}$

   $\Rightarrow 80\,^{\circ}\mathrm{F} = T_f + Ce^{-k(0\text{ min})} = T_f + C = -10\,^{\circ}\mathrm{F} + C$

   i.e., $80\,^{\circ}\mathrm{F} = -10\,^{\circ}\mathrm{F} + C$

   $\Rightarrow 90\,^{\circ}\mathrm{F} = C$

   $T = T_f + 90\,^{\circ}\mathrm{F}e^{-kt}$

   Because $T_f = -10\,^{\circ}\mathrm{F}$, this becomes:

   $T = -10\,^{\circ}\mathrm{F} + 90\,^{\circ}\mathrm{F}e^{-kt}$

   Also: At $T = 10$ minutes, $T = 60\,^{\circ}\mathrm{F}$

   $\Rightarrow 60\,^{\circ}\mathrm{F} = -10\,^{\circ}\mathrm{F} + 90\,^{\circ}\mathrm{F}e^{-k(10\text{ min})}$

   $\Rightarrow 70\,^{\circ}\mathrm{F} = 90\,^{\circ}\mathrm{F}e^{-k(10\text{ min})}$

   $\Rightarrow \frac{70\,^{\circ}\mathrm{F}}{90\,^{\circ}\mathrm{F}} = e^{-k(10\text{ min})}$

i.e., $\frac{7}{9} = e^{-k(10\text{ min})}$

$\Rightarrow \ln\left(\frac{7}{9}\right) = -k\,(10\text{ min})$

$\Rightarrow \frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}} = -k$

$\Rightarrow T = -10\,°\text{F} + 90\,°\text{F}e^{\left(\frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}}\right)t}$

At $T = 28\,°\text{F}$, we have:

$28\,°\text{F} = -10\,°\text{F} + 90\,°\text{F}e^{\left(\frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}}\right)t}$

$\Rightarrow 38\,°\text{F} = 90\,°\text{F}e^{\left(\frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}}\right)t}$

$\Rightarrow \frac{38\,°\text{F}}{90\,°\text{F}} = e^{\left(\frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}}\right)t}$

i.e., $\frac{3.8}{9} = e^{\left(\frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}}\right)t}$

$\Rightarrow \ln\left(\frac{3.8}{9}\right) = \left(\frac{\ln\left(\frac{7}{9}\right)}{10\text{ min}}\right)t$

$\Rightarrow \frac{\ln\left(\frac{3.8}{9}\right)(10\text{ min})}{\ln\left(\frac{7}{9}\right)} = t$

i.e., When $t = \frac{\ln\left(\frac{3.8}{9}\right)(10)}{\ln\left(\frac{7}{9}\right)}$ minutes, $T = 28\,°\text{F}$

2. The demand and supply of a certain commodity are given in terms of thousands of units, respectively, by:

$$D = 50 + 12p(t) + 2p'(t); \quad S = 450 - 8p(t) - 2p'(t).$$

At $t = 0$, the price of the commodity is 40 units.

a) Find the price at any later time and obtain its graph.

b) Determine whether there is price stability and the equilibrium price if one exists.

$D = 100 + 29p(t) + 7p'(t); \quad S = 900 - 11p(t) - p'(t).$

Setting the two equal, we have:

$8p'(t) + 40p(t) = 800$

We solve this equation using the " Integrating Factor Method."

$\Rightarrow p'(t) + 5p(t) = 100$

$\Rightarrow \frac{d}{dt}\left[e^{5t}p(t)\right] = 100e^{5t}$

$\Rightarrow \int \frac{d}{dt}\left[e^{5t}p(t)\right] dt = \int 100e^{5t} dt$

$\Rightarrow e^{5t}p(t) = 20e^{5t} + C$

$\Rightarrow p(t) = 20 + Ce^{-5t}$

Recall: At $t = 0$, the price of the commodity is 40 units.

$\Rightarrow 40 = 20 + Ce^{-5(0)}$

$\Rightarrow 20 = Ce^{-5(0)}$

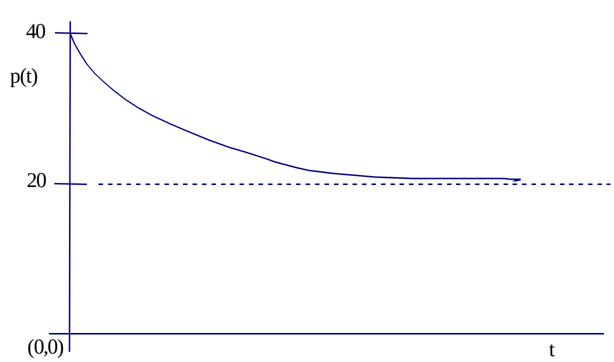
$\Rightarrow 20 = C$

$\Rightarrow p(t) = 20 + 20e^{-5t}$

a) $p(t) = 20 + 20e^{-5t}$

Note that $p'(t) = -100e^{-5t} < 0$.

Hence, $p(t)$ is decreasing for all $t > 0$



To determine stability, we consider the behavior of $p(t)$ as $t \to \infty$.

$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \left(20 + 20e^{-5t}\right) = 20$

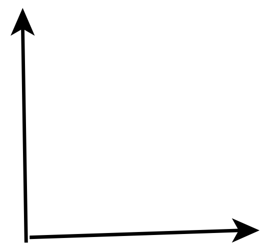
The price is **stable** as $p(t)$ tends to $p_e = 20$ units as $t \to \infty$.

3. The force of water resistance acting on a boat is proportional to its instantaneous velocity, and is such that at 30 $\frac{\text{ft}}{\text{sec}}$ the water resistance is 90 lb. If the boat and passengers combined weigh 960 lb, and if the motor exerts a steady force of 120 lb in the direction of the motion:

   (a) Find the velocity at any time $t \geq 0$, assuming that the boat starts from rest.

   (b) Find the limiting velocity

First of all, we will establish positive direction(s)

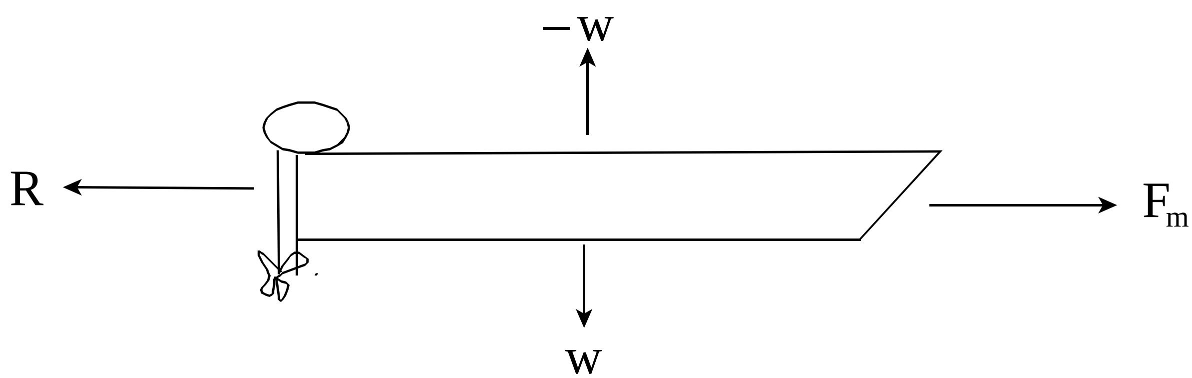


Positive Direction

Note: "The force of water resistance R is proportional to its instantaneous velocity v."

i.e., $R = kv$, where k is the constant of proportionality.

We draw a "force diagram" of the boat (below).



The total force acting on the boat *(in the horizontal direction)* is given by:

$F = R + F_m$, where F_m is the force applied by the motor.

We will let w be the combined weight of the boat and the passengers. The buoyant force acting on the boat is equal to $-w$.

(Note that the sum of the *vertical* forces acting on the boat is zero. Otherwise, the boat would either move upward or downward, depending on whether the sum of the vertical forces is positive or negative.)

The total (horizontal) force F is given by:

$F = R + F_m$

Here's the key to solving this type of exercise:

Recall that $F = ma$

By substituting ma for F, we get an equation in terms of velocity and acceleration:

$\underbrace{ma}_{F} = \underbrace{kv}_{R} + \underbrace{120 \text{ lb}}_{F_m}$

The importance of getting an equation in terms of velocity and acceleration, is that acceleration is the derivative of velocity (i.e., $a = \frac{dv}{dt}$).

Thus, by substituting $\frac{dv}{dt}$ for a, our equation becomes:

$m\frac{dv}{dt} = kv + 120$ lb., which is a differential equation in terms of velocity v.

We can solve the equation using the "Integrating Factor Method."

$\Rightarrow m\frac{dv}{dt} - kv = 120 \text{ lb}$

$\Rightarrow \frac{dv}{dt} - \frac{k}{m}v = \frac{120 \text{ lb}}{m}$

$\Rightarrow \frac{dv}{dt} + \underbrace{\left(-\frac{k}{m}\right)}_{P(t)} v = \frac{120 \text{ lb}}{m}$

Our integrating factor is $e^{\int p(t)dt} = e^{\int \left(-\frac{k}{m}\right)dt} = e^{-\frac{k}{m}t}$

Multiply both sides by the integrating factor

$\Rightarrow e^{-\frac{k}{m}t}\frac{dv}{dt} + \left(-\frac{k}{m}\right)e^{-\frac{k}{m}t}v = \frac{120 \text{ lb}}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right] = \frac{120 \text{ lb}}{m}e^{-\frac{k}{m}t}$

Integrate both sides with respect to t

$\Rightarrow \int \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right]dt = \frac{120 \text{ lb}}{m}\int e^{-\frac{k}{m}t}dt$

$\Rightarrow e^{-\frac{k}{m}t}v = \frac{120 \text{ lb}}{m}\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t} + C = -\frac{120 \text{ lb}}{k}e^{-\frac{k}{m}t} + C$

i.e., $e^{-\frac{k}{m}t}v = -\frac{120 \text{ lb}}{k}e^{-\frac{k}{m}t} + C$

Solve for v

$\Rightarrow v = -\frac{120 \text{ lb}}{k} + Ce^{\frac{k}{m}t}$

Next, we solve for the constant C, using our initial condition: "the boat starts from rest."

i.e. $v\,(0 \text{ sec}) = 0\frac{\text{ft}}{\text{sec}}$

$\Rightarrow 0\frac{\text{ft}}{\text{sec}} = v\,(0 \text{ sec}) = -\frac{120 \text{ lb}}{k} + Ce^{\frac{k}{m}(0 \text{ sec})} = -\frac{120 \text{ lb}}{k} + C$

i.e., $0\frac{\text{ft}}{\text{sec}} = -\frac{120 \text{ lb}}{k} + C$

$\Rightarrow C = \frac{120 \text{ lb}}{k}$

$\Rightarrow v = -\frac{120 \text{ lb}}{k} + \frac{120 \text{ lb}}{k} e^{\frac{k}{m}t}$

Let's find the other constants

$\boxed{k}$

Recall: $R = kv$

$\Rightarrow k = \frac{R}{v}$

Recall: at $v = 30\ \frac{\text{ft}}{\text{sec}}$, $R = -90$ lb

$\Rightarrow k = \frac{-90 \text{ lb}}{30\ \frac{\text{ft}}{\text{sec}}} = -3\frac{\text{lb sec}}{\text{ft}}$

$$\boxed{k = -3\frac{\text{lb sec}}{\text{ft}}}$$

$\boxed{m}$

Recall: $w = mg$

$\Rightarrow m = \frac{w}{g} = \frac{-960 \text{ lb}}{-32\ \frac{\text{ft}}{\text{sec}^2}} = 30\frac{\text{lb sec}^2}{\text{ft}}$

$$\boxed{m = 30\frac{\text{lb sec}^2}{\text{ft}}}$$

$$\Rightarrow v = -\frac{120 \text{ lb}}{\left(-3\frac{\text{lb sec}}{\text{ft}}\right)} + \frac{120 \text{ lb}}{\left(-3\frac{\text{lb sec}}{\text{ft}}\right)} e^{\frac{\left(-3\frac{\text{lb sec}}{\text{ft}}\right)}{\left(30\frac{\text{lb sec}^2}{\text{ft}}\right)}t} = 40\frac{\text{ft}}{\text{sec}} - 40\frac{\text{ft}}{\text{sec}} e^{-\frac{t}{10 \text{ sec}}}$$

$$\boxed{\text{i.e., } v\,(t) = 40\frac{\text{ft}}{\text{sec}} - 40\frac{\text{ft}}{\text{sec}} e^{-\frac{t}{10 \text{ sec}}}}$$

To find the limiting velocity, we let $t \to \infty$

$$\lim_{t\to\infty} v\,(t) = \lim_{t\to\infty} \left(40\frac{\text{ft}}{\text{sec}} - 40\frac{\text{ft}}{\text{sec}} e^{-\frac{t}{10 \text{ sec}}}\right) = 40\frac{\text{ft}}{\text{sec}}$$

$$\boxed{\text{i.e., } \lim_{t\to\infty} v\,(t) = 40\frac{\text{ft}}{\text{sec}}}$$