MTH 1125 Test #2 - Solutions

Summer 2021

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Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[4x^5+5x^4-6x^3+9x^2+10x+20\sqrt{x}+5\right]=$

Rewrite: $\frac{d}{dx}\left[4x^5+5x^4-6x^3+9x^2+10x+20x^{\frac{1}{2}}+5\right]$

$$=4\left(5x^4\right)+5\left(4x^3\right)-6\left(3x^2\right)+9\left(2x\right)+10+20\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)+0$$

$$=20x^4+20x^3-18x^2+18x+10+10x^{-\frac{1}{2}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[4x^5+5x^4-6x^3+9x^2+10x+20\sqrt{x}+5\right]=20x^4+20x^3-18x^2+18x+10+10x^{-\frac{1}{2}}}$$

2. Compute: $\frac{d}{dx}\left[\left(3x^4+2x+5\right)\sin(x)\right]=$

$$\frac{d}{dx}\left[\underbrace{\left(3x^4+2x+5\right)}_{1^{st}}\underbrace{\sin(x)}_{2^{nd}}\right]=\underbrace{\left(12x^3+2\right)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\sin(x)}_{2^{nd}}+\underbrace{\cos(x)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\left(3x^4+2x+5\right)}_{1^{st}}$$

$$\boxed{\frac{d}{dx}\left[\left(3x^4+2x+5\right)\sin(x)\right]=\left(12x^3+2\right)\sin(x)+\cos(x)\left(3x^4+2x+5\right)}$$

3. Compute: $\frac{d}{dx}\left[\frac{4x^3+6x^2+12x+2}{8x^5+5x}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{4x^3+6x^2+12x+2}^{\text{top}}}{\underbrace{8x^5+5x}_{\text{Bottom}}}\right]=\frac{\overbrace{\left(12x^2+12x+12\right)}^{\text{top prime}}\cdot\overbrace{\left(8x^5+5x\right)}^{\text{bottom}}-\overbrace{\left(40x^4+5\right)}^{\text{bottom prime}}\cdot\overbrace{\left(4x^3+6x^2+12x+2\right)}^{\text{top}}}{\underbrace{\left(8x^5+5x\right)^2}_{\text{bottom squared}}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\frac{4x^3+6x^2+12x+2}{8x^5+5x}\right]=\frac{\left(12x^2+12x+12\right)\left(8x^5+5x\right)-\left(40x^4+5\right)\left(4x^3+6x^2+12x+2\right)}{\left(8x^5+5x\right)^2}}$$

4. Compute: $\frac{d}{dx}\left[\left(8x^4+8x^2+5\right)^{10}\right]=$ This is the derivative of a function raised to a power.

$$\frac{d}{dx}\left[\left(8x^4+8x^2+5\right)^{10}\right]=\underbrace{10\left(8x^4+8x^2+5\right)^9}_{\substack{\text{power rule}\\ \text{as usual}}}\cdot\underbrace{\left(32x^3+16x\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\left(8x^4+8x^2+5\right)^{10}\right]=10\left(8x^4+8x^2+5\right)^9\left(32x^3+16x\right)}$$

5. Given that $f(x)=3x^2+2x+4$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(2,20)$.

We need two things:

i. A **point** on the line (We have that: $(x_1,y_1)=(2,20)$)

ii. The **slope** of the line (This is $f'(x_1)$)

$f'(x)=6x+2$

At the point $(x_1,y_1)=(2,20)$, **the slope is** $f'(2)=6(2)+2=14$

We will use the Point-Slope equation of a line:

$(y-y_1)=m(x-x_1)$ (Where m is the slope and (x_1,y_1) is a known point on the line.)

Thus, the equation of the line tangent to the graph of $f(x)$ is:

$(y-20)=14(x-2)$

$$\boxed{\text{The equation of the line tangent is } (y-20)=14(x-2)}$$

6. Given that $t = 3x^2 + 2x + 5$ and that $x = \sin(w)$; compute $\frac{dt}{dw}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

We know:

$\frac{dt}{dx} = 6x + 2$

$\frac{dx}{dw} = \cos(w)$

We want: $\frac{dt}{dw}$

By the Liebniz form of the Chain Rule:

$$\frac{dt}{dw} = \frac{dt}{dx}\frac{dx}{dw} = (6x+2)\cos(w) = \underbrace{(6\sin(w)+2)\cos(w)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } v}}$$

i.e. $\frac{dt}{dw} = (6\sin(w)+2)\cos(w)$

7. Compute: $\frac{d}{dx}\left[\tan\left(4x^2 - 3x + 2\right)\right] =$

Outer:	=	sec ( )
Deriv. of outer	=	sec ( ) tan ( )

$$\frac{d}{dx}\left[\begin{array}{l} \tan\left(\underbrace{4x^2-3x+2}\right) \\ \uparrow \qquad\quad \uparrow \\ \text{outer} \quad \text{inner} \end{array}\right] = \underbrace{\sec^2\left(4x^2-3x+2\right)}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{(8x-3)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\tan\left(4x^2-3x+2\right)\right] = \sec^2\left(4x^2-3x+2\right)(8x-3)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{5x^4+10x^2+5}{8x^3+12x^2}\right)^{10}\right] =$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE.

$$\frac{d}{dx}\left[\underbrace{\left(\frac{5x^4+10x^2+5}{8x^3+12x^2}\right)^{10}}_{(g(x))^n}\right] = \underbrace{10\left(\frac{5x^4+10x^2+5}{8x^3+12x^2}\right)^{9}}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{5x^4+10x^2+5}{8x^3+12x^2}\right]\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

$$= 10\left(\frac{5x^4+10x^2+5}{8x^3+12x^2}\right)^{9} \underbrace{\frac{(20x^3+20x)(8x^3+12x^2) - (24x^2+24x)(5x^4+10x^2+5)}{(8x^3+12x^2)^2}}_{\substack{\text{quotient}\\ \text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{5x^4+10x^2+5}{8x^3+12x^2}\right)^{10}\right] = 10\left(\frac{5x^4+10x^2+5}{8x^3+12x^2}\right)^{9} \frac{\left(20x^3+20x\right)\left(8x^3+12x^2\right) - \left(24x^2+24x\right)\left(5x^4+10x^2+5\right)}{\left(8x^3+12x^2\right)^2}$

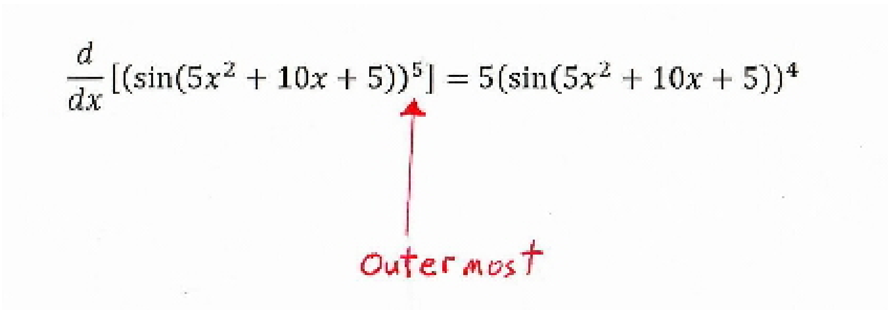
9. Compute: $\frac{d}{dx}\left[\sin^5\left(5x^2+10x+5\right)\right] =$

Let's rewrite this:

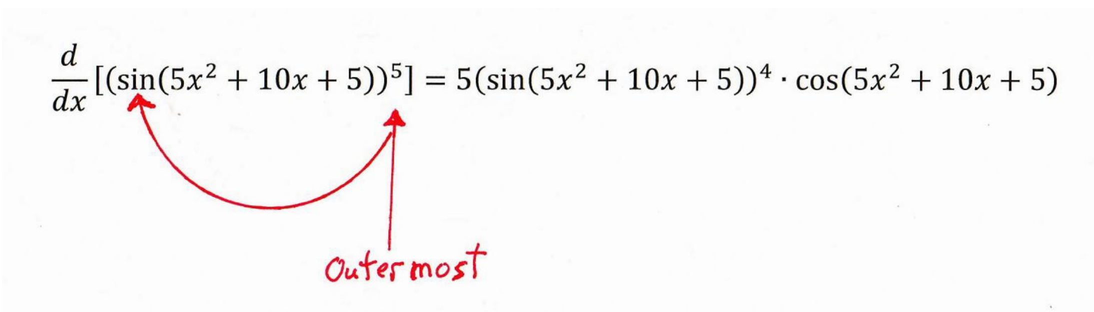
$\frac{d}{dx}\left[\left(\sin\left(5x^2+10x+5\right)\right)^5\right]$

This is the composition of *three* functions.

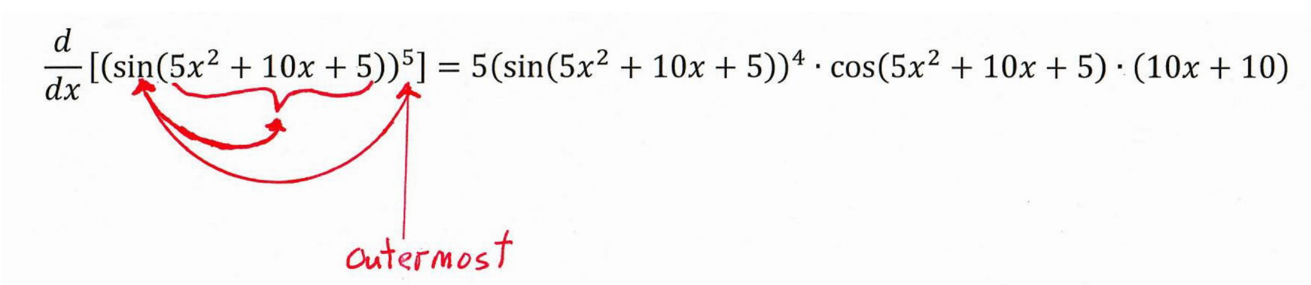
Differentiate the outermost function and evaluate it at everything inside



Next: Multiply by the derivative of the next outermost function and evaluate it at everything inside of it.



Finally: Multiply by the derivative of the innermost function.



i.e., $\frac{d}{dx}\left[\sin^5\left(5x^2+10x+5\right)\right] = 5\left(\sin\left(5x^2+10x+5\right)\right)^4\cos\left(5x^2+10x+5\right)\cdot\left(10x+10\right)$

Alternatively:

$\frac{d}{dx}\left[\sin^5\left(5x^2+10x+5\right)\right]$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE. Re-write!

$\frac{d}{dx}\left[\left(\sin\left(5x^2+10x+5\right)\right)^5\right]$ This is the derivative of a function, raised to a power

$$\frac{d}{dx}\left[\left(\sin\left(5x^2+10x+5\right)\right)^5\right] = \underbrace{5\left(\sin\left(5x^2+10x+5\right)\right)^4}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\sin\left(5x^2+10x+5\right)\right]\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$= 5\left(\sin\left(5x^2+10x+5\right)\right)^4 \cdot \underbrace{\left[\cos\left(5x^2+10x+5\right)\cdot\left(10x+10\right)\right]}_{\substack{\text{Chain}\\ \text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\sin^5\left(5x^2+10x+5\right)\right] = 5\left(\sin\left(5x^2+10x+5\right)\right)^4\cos\left(5x^2+10x+5\right)\cdot\left(10x+10\right)$

10. Given that $3x^4 + 8x^3y^3 + 5y^5 = \cos(y)$, compute $\frac{dy}{dx}$

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[3x^4 + \underbrace{8x^3}_{1^{\text{st}}}\underbrace{y^3}_{2^{\text{nd}}} + 5y^5\right] = \frac{d}{dx}\left[\cos(y)\right]$$

$$\Rightarrow 12x^3 + \left(\underbrace{24x^2}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^3}_{2^{\text{nd}}} + \underbrace{3y^2\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{8x^3}_{1^{\text{st}}} + 25y^4\frac{dy}{dx}\right) = -\sin(y)\cdot\frac{dy}{dx}$$

Simplifying, we have:

$$12x^3 + 24x^2y^3 + 24x^3y^2\frac{dy}{dx} + 25y^4\frac{dy}{dx} = -\sin(y)\frac{dy}{dx}$$

ii. Solve algebraically for $\frac{dy}{dx}$

a. Get $\frac{dy}{dx}$ terms on left side, all other terms on right side

$$\Rightarrow 24x^3y^2\frac{dy}{dx} + 25y^4\frac{dy}{dx} + \sin(y)\frac{dy}{dx} = -12x^3 - 24x^2y^3$$

b. Factor out $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}\left(24x^3y^2 + 25y^4 + \sin(y)\right) = -12x^3 - 24x^2y^3$$

c. Divide both sides by the cofactor of $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-12x^3-24x^2y^3}{24x^3y^2+25y^4+\sin(y)} = -\frac{12x^3+24x^2y^3}{24x^3y^2+25y^4+\sin(y)}$$

$$\boxed{\frac{dy}{dx} = \frac{-12x^3-24x^2y^3}{24x^3y^2+25y^4+\sin(y)} = -\frac{12x^3+24x^2y^3}{24x^3y^2+25y^4+\sin(y)}}$$

11. Given that $f(x) = x^2 + 4x + 2$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^2 + 4(x+\Delta x) + 2\right] - \left[x^2+4x+2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[\left(x^2 + 2x\Delta x + \Delta x^2\right) + 4(x+\Delta x) + 2\right] - \left[x^2+4x+2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[x^2 + 2x\Delta x + \Delta x^2 + 4x + 4\Delta x + 2\right] - \left[x^2+4x+2\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 + 4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x + 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x + 4) = 2x + (0) + 4 = 2x + 4$$

$$\boxed{\text{i.e., } f'(x) = 2x + 4}$$

Extra (Wow! 10 Points)

Given that $T'(x) = \frac{1}{1+x^2}$ (i.e., $\frac{d}{dx}\left[T(x)\right] = \frac{1}{1+x^2}$); compute $\frac{d}{dx}\left[T(\sin(x))\right]$

Outer:	$=$	$T(\quad)$
Deriv. of outer	$=$	$\frac{1}{1+(\quad)^2}$

$$\frac{d}{dx}\Big[\underset{\uparrow \text{ outer}}{T}\Big(\underset{\uparrow \text{ inner}}{\underbrace{\sin(x)}}\Big)\Big] = \underbrace{\frac{1}{1+(\sin(x))^2}}_{\text{Deriv of outer, eval. at inner}} \cdot \underbrace{\cos(x)}_{\text{deriv. of inner}} = \frac{\cos(x)}{1+(\sin(x))^2}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[T(\sin())\right] = \frac{1}{1+(\sin(x))^2} \cdot \cos(x) = \frac{\cos(x)}{1+(\sin(x))^2}}$$