Definitions, Theorems, etc. Part #1

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Definition 1 *Given a real-valued function* $f(x)$, *the statement:* $\lim_{x\to c} f(x) = L$ *means that given* $\varepsilon > 0$, $\exists\delta = \delta(\varepsilon) > 0$ *such that* $0 < |x - c| < \delta \Rightarrow |f(x) = L| < \varepsilon$.

Theorem 2 $\lim_{x\to c} f(x)$ *(if such a limit exists) is unique.*

Theorem 3 *Given real-valued functions* $f(x)$ *and* $g(x)$, *such that* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$:

a) $\lim_{x\to c}(f(x) + g(x)) = L + M$

b) $\lim_{x\to c}(f(x) - g(x)) = L - M$

c) $\lim_{x\to c} f(x)\,g(x) = LM$

d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$; *provided that* $M \neq 0$

Theorem 4 $\lim_{x\to c} k = k$, *for any constant* k.

Theorem 5 *If* $\lim_{x\to c} f(x) = L$, *then* $\lim_{x\to c} kf(x) = kL$, *for any constant* k.

Definition 6 *Given a real-valued function* $f(x)$, *the statement:* $\lim_{x\to c} f(x) = \infty$ *means that given* $M \in \mathbb{R}^+$, $\exists\delta = \delta(M) > 0$ *such that* $0 < |x - c| < \delta \Rightarrow f(x) > M$

Remark 7 $\lim_{x\to c} f(x) = -\infty$ *is defined analogously*

Definition 8 *Given a real-valued function* $f(x)$, *the statement:* $\lim_{x\to\infty} f(x) = L$ *means that given* $\varepsilon > 0$, $\exists N \in \mathbb{R}^+$ *such that* $x > N \Rightarrow |f(x) - L| < \varepsilon$

Remark 9 $\lim_{x\to -\infty} f(x) = L$ *is defined analogously*

Definition 10 *Given a real-valued function* $f(x)$, *the statement:* $\lim_{x\to\infty} f(x) = \infty$ *means that given* $M \in \mathbb{R}^+$, $\exists N \in \mathbb{R}^+$ *such that* $x > N \Rightarrow f(x) > M$

Remark 11 $\lim_{x\to\pm\infty} f(x) = \pm\infty$ *is defined analogously*

Definition 12 *A* ***sequence*** *of real numbers is the range of a function* $f : \mathbb{N} \to \mathbb{R}$, *such that the elements of the range inherit a well ordering induced by the elements of* $\mathbb{N}$.

For example, consider the function $f : \mathbb{N} \to \mathbb{R}$, *shown below:*

$$\begin{array}{rcccccccc} \mathbb{N} & = & \{1, & 2, & 3, & \ldots, & n, & \ldots & \} \\ f\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \\ \mathbb{R} & = & \{f(1), & f(2), & f(3), & \ldots, & f(n), & \ldots & \} \end{array}$$

$f(1)$ *is the* ***first term*** *of the sequence,* $f(2)$ *is the* ***second term*** *of the sequence,* $f(n)$ *is the* $\boldsymbol{n^{th}}$ ***term*** *of the sequence, etc.*

Example 13 *For example, given the function* $f : \mathbb{N} \to \mathbb{R}$, *defined by* $f(n) = \frac{1}{n}$, *the range is* $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$.

1 *is the* ***first term*** *of the sequence,* $\frac{1}{2}$ *is the* ***second term*** *of the sequence,* $\frac{1}{n}$ *is the* $\boldsymbol{n}^{th}$ ***term*** *of the sequence, etc.*

Notation 14 *By convention, we use subscripted, lower-case letters to denote the terms of a generic sequence as follows:*

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

By convention, we denote an entire generic sequence as follows:

$$\{a_n\}_{n=1}^{\infty}$$

$$\text{i.e., } \{a_n\}_{n=1}^{\infty} \text{ is the same as } \{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

Definition 15 *The sequence* $\{a_n\}_{n=1}^{\infty}$ ***converges*** *exactly when* $\exists\, L \in \mathbb{R}$ *such that* $\forall \varepsilon > 0$, $\exists\, N = N(\varepsilon) \in \mathbb{N}$, *such that* $n > N \Rightarrow |a_n - L| < \varepsilon$.

In this case, L *is said to be the* ***limit*** *of the sequence, and the sequence is said to converge to* L.

If no such real number L *exists, the sequence is said to* ***diverge.***

Definition 16 *Given the sequence* $\{a_n\}_{n=1}^{\infty}$, *the statement:* $\lim_{n\to\infty} a_n = \infty$ *means that given* $M \in \mathbb{R}^+$, $\exists N = N(M) \in \mathbb{N}$ *such that* $n > N \Rightarrow a_n > M$.

In this case, the sequence is said to ***diverge.***

Remark 17 *The statement:* $\lim_{n\to\infty} a_n = -\infty$ *is defined analogously, and this case, the sequence is said to* ***diverge.***

Theorem 18 *The limit of a convergent sequence is unique.*

Theorem 19 *If the sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to* L *and the sequence* $\{b_n\}_{n=1}^{\infty}$ *converges to* M, *then:*

a) the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ *converges to* $L + M$

a) the sequence $\{a_n - b_n\}_{n=1}^{\infty}$ *converges to* $L - M$

a) the sequence $\{a_n b_n\}_{n=1}^{\infty}$ *converges to* LM

a) the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *converges to* $\frac{L}{M}$; *provided that* $M \neq 0$

Theorem 20 *(Equivalent criterion for showing that a sequence converges) A sequence* $\{a_n\}_{n=1}^{\infty}$ ***converges*** *if to a limit* L *if and only if* $\forall \varepsilon > 0$, *the interval* $(L - \varepsilon, L + \varepsilon)$ ***contains all but finitely many terms*** *of the sequence.*

Definition 21 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers is* ***bounded above*** *if* $\exists\ M \in \mathbb{R}$ *such that* $a_n \leq M$ $\forall n \in \mathbb{N}$. *In this case,* M *is called an* ***upper bound*** *of* S.

Definition 22 *Analogously, a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers is* ***bounded below*** *if* $\exists\ m \in \mathbb{R}$ *such that* $a_n \geq m$ $\forall n \in \mathbb{N}$. *In this case,* m *is called a* ***lower bound*** *of* S.

Definition 23 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers is said to be* ***bounded*** *if it is bounded above and bounded below.*

Definition 24 *The real number* $\mathbf{U}$ *is the* ***least upper bound*** *(l.u.b.) of a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers exactly when:*

1) $\mathbf{U}$ *is an upper bound of* $\{a_n\}_{n=1}^{\infty}$, *and*

2) $\mathbf{U} \leq \mathbf{M}$ *for all other upper bounds* M *of* $\{a_n\}_{n=1}^{\infty}$.

Analogously, The real number $\mathbf{L}$ *is the* ***greatest lower bound*** *(g.l.b.) of a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers exactly when:*

1) $\mathbf{L}$ *is a lower bound of* $\{a_n\}_{n=1}^{\infty}$, *and*

2) $\mathbf{L} \geq \mathbf{m}$ *for all other lower bounds* m *of* $\{a_n\}_{n=1}^{\infty}$.

Axiom 25 *(Least Upper Bound Axiom for Sequences) Every sequence of real numbers that is bounded above has a least upper bound* $\mathbf{U}$. *(where* $\mathbf{U}$ *is also a real number).*

Analogously, Every sequence of real numbers that is bounded below has a greatest lower bound $\mathbf{L}$.

Theorem 26 *Every convergent sequence of real numbers is bounded.*

Definition 27 *A non-empty set of real numbers* S *is* ***bounded above*** *if* $\exists\ M \in \mathbb{R}$ *such that* $s \leq M$ $\forall s \in S$. *In this case,* M *is called an* ***upper bound*** *of* S.

Analogously, a non-empty set of real numbers S *is* ***bounded below*** *if* $\exists\ m \in \mathbb{R}$ *such that* $s \geq m$ $\forall s \in S$. *In this case,* m *is called a* ***lower bound*** *of* S.

Definition 28 *A non-empty set of real numbers* S *is said to be* ***bounded*** *if it is bounded below and bounded above.*

Definition 29 *The real number* $\mathbf{U}$ *is the* ***least upper bound*** *(l.u.b.) of a non-empty set of real numbers* S *exactly when:*

1) $\mathbf{U}$ *is an upper bound of* S, *and*

2) $\mathbf{U} \leq \mathbf{M}$ *for all other upper bounds* M *of* S.

Analogously, The real number $\mathbf{L}$ *is the* ***greatest lower bound*** *(g.l.b.) of a non-empty set of real numbers* S *exactly when:*

1) $\mathbf{L}$ *is a lower bound of* S, *and*

2) $\mathbf{L} \geq \mathbf{M}$ *for all other lower bounds* m *of* S.

Axiom 30 *(Least Upper Bound Axiom of Real Numbers) Every non-empty set of real numbers that is bounded above has a least upper bound* **U** *(where* **U** *is also a real number).*

Analogously, Every non-empty set of real numbers that is bounded below has a greatest lower bound **L** *(where* **L** *is also a real number).*

Example 31 *Consider the sequence* $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\right\}_{n=1}^{\infty}$

The real numbers $-3, -\frac{16}{5}, -1, \ldots$ *etc. are all lower bounds of the sequence.*

The real number 0 *is the greatest lower bound. First of all,* 0 *IS a lower bound of the sequence, since* $0 \leq \frac{n-1}{n}, \forall n \in \mathbb{N}$*). Second, no real number* $m > 0$ *can be a lower bound of the sequence, because it won't be less than or equal to the first term of the sequence,* 0*. Thus,* $0 \geq m$ *for all lower bounds* m*.*

The real number 1 *is the least upper bound. First of all,* 1 *IS an upper bound of the sequence, since* $1 > \frac{n-1}{n}$*. (The numerator is always less than the denominator.) Second, for any real number* x*, with* $0 < x < 1$*,* x *cannot be an upper bound for the sequence, because there will be terms of the sequence greater than* x*.*

To see this, let $0 < x < 1$*, and let* $\varepsilon = 1 - x$*. (i.e.* $x = 1 - \varepsilon$*)*

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Scratchwork: *We want* $\frac{n-1}{n} > x$

$\Leftrightarrow \frac{n-1}{n} > 1 - \varepsilon \Leftrightarrow 1 - \frac{1}{n} > 1 - \varepsilon \Leftrightarrow -\frac{1}{n} > -\varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$

i.e., just choose $n > \frac{1}{\varepsilon}$

End of Scratch

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Let $n > \frac{1}{\varepsilon}$

$\Leftrightarrow \varepsilon > \frac{1}{n} \Leftrightarrow -\varepsilon < -\frac{1}{n} \Leftrightarrow \underbrace{1 - \varepsilon}_{x} < \underbrace{1 - \frac{1}{n}}_{\frac{n-1}{n}} \Leftrightarrow x < \frac{n-1}{n}$

Thus, any real number $x < 1$ *cannot be an upper bound for* $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$

Thus, 1 *is the least upper bound of* $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$

Definition 32 *A sequence of real numbers* $\{a_n\}_{n=1}^{\infty}$ *is said to be* ***monotone increasing*** *if*

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$$

$\{a_n\}_{n=1}^{\infty}$ *is said to be* ***strictly monotone increasing*** *if*

$$a_1 < a_2 < a_3 < \ldots < a_n < \ldots$$

Analogously, $\{a_n\}_{n=1}^{\infty}$ *is said to be* ***monotone decreasing*** *if*

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$$

$\{a_n\}_{n=1}^{\infty}$ *is said to be* ***strictly monotone decreasing*** *if*

$$a_1 > a_2 > a_3 > \ldots > a_n > \ldots$$

Example 33 $\sqrt{2} = 1.4142136\ldots$

Consider the sequence $\{a_n\}_{n=1}^{\infty} = \{1.0, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142136, \ldots\}$

The sequence is bounded above by $\sqrt{2}$.

Intuitively, $\sqrt{2}$ *is the least upper bound.*

Note that as a sequence of ***rational numbers,*** $\{a_n\}_{n=1}^{\infty}$ *has a least upper bound* $\sqrt{2}$ *that is NOT a rational number.*

But as a sequence of ***real numbers,*** $\{a_n\}_{n=1}^{\infty}$ *has a least upper bound* $\sqrt{2}$ *that IS a real number.*

This is the sense of the Least Upper Bound Axiom of Real Numbers. Every sequence of real numbers, that is bounded above, converges (to a number that is also a real number).

Our example shows that we can't make an analogous statement about sequences of ***rational*** *numbers, that are bounded above.*

Theorem 34 *The Monotone Convergence Theorem (a.k.a. The Bounded Convergence Theorem) Every monotone increasing sequence of real numbers, that is bounded above, converges.*

(Similarly, every monotone decreasing sequence of real numbers, that is bounded below, converges.)

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Definition 35 *A sequence of intervals*

$$[a_1, b_1], [a_2, b_2], [a_3, b_3], \ldots, [a_n, b_n], \ldots$$

is said to be a ***sequence of nested intervals*** *(or is said to satisfy the* ***nested interval property****) if the following two conditions hold:*

i) $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]\ \forall n \in \mathbb{N}$

ii) $\lim_{n\to\infty}(b_n - a_n) = 0$

Theorem 36 *(Nested Interval Theorem) Suppose that* $\{[a_n, b_n]\}_{n=1}^{\infty} = [a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots$ *is a sequence of nested intervals. Then* $\exists!$ *point* p *such that* $p \in \cap_{n=1}^{\infty}[a_n, b_n]$

Definition 37 *A point* p *is a* ***limit point of a sequence*** $\{a_n\}_{n=1}^{\infty}$, *if* $\forall\varepsilon > 0$, *the interval* $(p - \varepsilon, p + \varepsilon)$ *contains infinitely many terms of the sequence.*

Definition 38 *A point* p *is a* ***limit point of a set of real numbers*** S, *if* $\forall\varepsilon > 0$, *the interval* $(p - \varepsilon, p + \varepsilon)$ *contains a point of* S *other than* p.

Theorem 39 *A point* p *is a* ***limit point of a set of real numbers*** S, *if and only if* $\forall\varepsilon > 0$, *the interval* $(p - \varepsilon, p + \varepsilon)$ *contains infinitely many points of* S.

Theorem 40 *The limit* L *of a sequence is a limit point of the sequence.*

Theorem 41 *A convergent sequence can have no limit points other than the limit* L.

Theorem 42 *(Bolzano-Weierstrass) Every bounded set containing infinitely many real numbers has a limit point.*

Theorem 43 *(Bolzano-Weierstrass Theorem for Sequences) Every bounded sequence has a limit point.*

Definition 44 *A sequence* $\{b_1, b_2, b_3, \ldots, b_n, \ldots\}$ *is a* ***subsequence*** *of the sequence* $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ *if the following two conditions hold:*

i) $\forall n \in \mathbb{N}$, b_n *is a term of the sequence* $\{a_n\}_{n=1}^{\infty}$

ii) The relative order of the terms of the sequence $\{b_n\}_{n=1}^{\infty}$ *is the same as the relative order of the same terms as elements of the sequence* $\{a_n\}_{n=1}^{\infty}$.

Theorem 45 *Every bounded sequence has a convergent subsequence.*

Theorem 46 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to* L *if and only if every subsequence of* $\{a_n\}_{n=1}^{\infty}$ *converges to* L.

Definition 47 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *is a* ***Cauchy Sequence*** *if* $\forall\varepsilon > 0$, $\exists\ N = N(\varepsilon) \in \mathbb{N}$ *such that* $m, n > N \Rightarrow |a_m - a_n| < \varepsilon$.

Theorem 48 *Every convergent sequence is a Cauchy Sequence.*

Theorem 49 *Every Cauchy Sequence has a limit point.*

Theorem 50 *Every Cauchy Sequence is convergent.*

Corollary 51 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *converges if and only it is a Cauchy Sequence.*

Definition 52 *The real number* L *is the* ***limit*** *of a function* $f(x)$, *as* x *approaches* c, *exactly when:*

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ \text{such that } 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

In this case, we write:

$$\lim_{x \to c} f(x) = L$$

Definition 53 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is* ***continuous at a point*** x_0 *exactly when:*

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Definition 54 *(Alternate) A function* $f : \mathbb{R} \to \mathbb{R}$ *is* ***continuous at a point*** x_0 *exactly when:*

$$\lim_{\Delta x \to 0} f(x_0 + \Delta x) = f(x_0)$$

Definition 55 *(Alternate) A function* $f : \mathbb{R} \to \mathbb{R}$ *is* ***continuous at a point*** x_0 *exactly when:* $\forall \varepsilon > 0$, $\exists$ $\delta = \delta(\varepsilon, x_0)$ *such that*

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Definition 56 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is* ***continuous*** *if it is continuous at each point* $x \in \mathbb{R}$

Similarly, given $B \subseteq \mathbb{R}$, *a function* $f : B \to \mathbb{R}$ *is* ***continuous*** *if it is continuous at each point* $x \in B$

Theorem 57 *The constant function* $f(x) = c$ *is continuous*

Theorem 58 *Sum, differences, and products of continuous functions are continuous*

Theorem 59 *The function* $f(x) = x$ *is continuous.*

Theorem 60 *The function* $f(x) = x^n$ *is continuous for* $n = 0, 1, 2, 3, \ldots$

Definition 61 *A **monomial** in the variable x is an expression of the form cx^n, where $c \in \mathbb{R}$ and n is a non-negative integer.*

Definition 62 *A **polynomial** in the variable x is the sum of one or more monomials in the variable x.*

Theorem 63 *All monomials are continuous*

Theorem 64 *All polynomials are continuous*

Theorem 65 *Suppose that $f(x)$ and $g(x)$ are defined for all $x \in [a,b]$, except possibly for some point $c \in (a.b)$.*

Suppose also that $f(x) \leq g(x)$ for all $x \in [a,b]$, except possibly for some point $c \in (a.b)$.

Finally, suppose that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, for some $L, M \in \mathbb{R}$.

Then $L \leq M$. (i.e., $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$)

Theorem 66 *(Sandwich Theorem) Suppose that $f(x), g(x)$, and $h(x)$ are defined for all $x \in [a,b]$, except possibly for some point $c \in (a.b)$.*

Suppose also that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$, except possibly for some point $c \in (a.b)$.

Finally, suppose that $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$.

Then $\lim_{x \to c} g(x) = L$ also.

Theorem 67 $\lim_{x \to 0} \sin(x) = 0$. *($x$ is measured in radians)*

Theorem 68 $\lim_{x \to 0} \cos(x) = 1$. *($x$ is measured in radians)*

Theorem 69 *The function $f(x) = \sin(x)$ is continuous on $(-\infty, \infty)$.*

Theorem 70 *The function $f(x) = \cos(x)$ is continuous on $(-\infty, \infty)$.*

Theorem 71 *A real-valued function $f(x)$ is continuous at a point $x = L$ if and only if the sequence $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(L)$, whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to L.*