MTH 3311 - Practice Test #3 - Solutions

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Show CLEARLY how you arrive at you answers!

1. Find the general solution to the equation $x^2y'' + 5xy' + 4y = 5x^4 + 3x^{\frac{1}{2}}$

   First, find the solution to the complementary equation $x^2y'' + 5xy' + 4y = 0$

   Our strategy is to seek solutions of the form:

   $y = x^\lambda$

   $\Rightarrow y' = \lambda x^{\lambda-1}$

   $\Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2}$

   Plugging these into the complementary equation $x^2y'' + 5xy' + 4y = 0$, we have:

   $x^2\lambda(\lambda-1)x^{\lambda-2} + 5x\lambda x^{\lambda-1} + 4x^\lambda = 0$

   $\Rightarrow \lambda(\lambda-1)x^\lambda + 5\lambda x^\lambda + 4x^\lambda = 0$

   $\Rightarrow \lambda(\lambda-1) + 5\lambda + 4 = 0$

   $\Rightarrow \lambda^2 + 4\lambda + 4 = 0$

   $\Rightarrow (\lambda+2)^2 = 0$

   $\Rightarrow \lambda_1 = \lambda_2 = -2$; (Double Root)

   Our complementary solution *should* be of the form:

   $y_c = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^{-2} + c_2x^{-2}$

   Oops! c_1x^{-2} and c_2x^{-2} are not independent solutions - they are the *same* solution.

   To find another independent solution, we multiply by $\ln(x)$.

   (This is one way in which our approach for Euler Equations differs from our approach for Equations with Constant Coefficients. With Constant Coefficients, we multiply one of the terms by x to get another independent solution. With Euler Equations, we multiply by $\ln(x)$.)

   So our complementary solution is:

   $y_c = c_1x^{-2} + c_2\ln(x)x^{-2}$

Next, we find our particular solution

Since the right hand side of the equation is a linear combination of powers of x, the Method of Undetermined Coefficients will work. This is another way in which Euler Equations differ from Equations with Constant Coefficients.

With *Equations with Constant Coefficients,* The Method of Undetermined Coefficients works whenever the right hand side is a linear combination of sines and cosines, and/or exponential functions, and/or polynomials.

With *Euler Equations,* The Method of Undetermined Coefficients works whenever the right hand side is any linear combination of powers of x - and that includes negative powers of x and fractional powers of x. The Method of Undetermined Coefficients WILL NOT work if the right hand side contains trig functions or exponentials.

Since the right hand side is $5x^4 + 3x^{\frac{1}{2}}$, we guess that the particular solution is a linear combination of exactly those powers of x that appear on the right hand side of the equation:

$$y = Ax^4 + Bx^{\frac{1}{2}}$$

$\Rightarrow y' = 4Ax^3 + \frac{1}{2}Bx^{-\frac{1}{2}}$

$\Rightarrow y'' = 12Ax^2 - \frac{1}{4}Bx^{-\frac{3}{2}}$

Plugging these in the equation: $x^2y'' + 5xy' + 4y = 5x^4 + 3x^{\frac{1}{2}}$, we have:

$$x^2 \underbrace{\left(12Ax^2 - \frac{1}{4}Bx^{-\frac{3}{2}}\right)}_{y''} + 5x \underbrace{\left(4Ax^3 + \frac{1}{2}Bx^{-\frac{1}{2}}\right)}_{y'} + 4 \underbrace{\left(Ax^4 + Bx^{\frac{1}{2}}\right)}_{y} = 5x^4 + 3x^{\frac{1}{2}}$$

$\Rightarrow \left(12Ax^4 - \frac{1}{4}Bx^{\frac{1}{2}}\right) + 5\left(4Ax^4 + \frac{1}{2}Bx^{\frac{1}{2}}\right) + 4\left(Ax^4 + Bx^{\frac{1}{2}}\right) = 5x^4 + 3x^{\frac{1}{2}}$

$\Rightarrow (12 + 20 + 4)\, Ax^4 + \left(-\frac{1}{4} + \frac{5}{2} + 4\right) Bx^{\frac{1}{2}} = 5x^4 + 3x^{\frac{1}{2}}$

$36Ax^4 + \frac{25}{4}Bx^{\frac{1}{2}} = 5x^4 + 3x^{\frac{1}{2}}$

Equating Coefficients of like powers of x, we have:

$36A = 5 \Rightarrow A = \frac{5}{36}$

$\frac{25}{4}B = 3 \Rightarrow B = \frac{12}{25}$

$y_p = Ax^4 + Bx^{\frac{1}{2}} = \frac{5}{36}x^4 + \frac{12}{25}x^{\frac{1}{2}}$

Our general solution is $y = y_p + y_c$

$y = \frac{5}{36}x^4 + \frac{12}{25}x^{\frac{1}{2}} + c_1x^{-2} + c_2\ln(x)\,x^{-2}$

$$\boxed{y = \frac{5}{36}x^4 + \frac{12}{25}x^{\frac{1}{2}} + c_1 x^{-2} + c_2 \ln(x)\, x^{-2}}$$

2. Find the general solution to the equation $x^2y'' + 5xy' + 4y = 5x^{-2}$

From the previous exercise, the complementary solution is:

$y_c = c_1x^{-2} + c_2 \ln(x)\, x^{-2}$

Since the right hand side of the equation is $5x^{-2}$, we guess that our particular solution is Thus, we guess that:

$y_p = Ax^{-2}$ Oops! This is one of the independent terms of our complementary solution.

So, we multiply by $\ln(x)$, which yields:

$y_p = A\ln(x)\, x^{-2}$ Oops! This is the other independent term of our complementary solution.

What do we do now? You guessed it – multiply by $\ln(x)$ **again!** This yields:

$y_p = A(\ln(x))^2 x^{-2}$

$\Rightarrow y' = 2A\ln(x)\frac{1}{x}x^{-2} - 2x^{-3}A(\ln(x))^2 = 2A\ln(x)\, x^{-3} - 2x^{-3}A(\ln(x))^2$

i.e., $y' = 2A\ln(x)\, x^{-3} - 2x^{-3}A(\ln(x))^2$

$\Rightarrow y'' = 2A\frac{1}{x}x^{-3} - 6Ax^{-4}\ln(x) + 6x^{-4}A(\ln(x))^2 + 2\ln(x)\frac{1}{x}(-2Ax^{-3})$

i.e., $y'' = 2Ax^{-4} - 6Ax^{-4}\ln(x) + 6x^{-4}A(\ln(x))^2 - 4A\ln(x)\, x^{-4}$

Plugging these into the equation $x^2y'' + 5xy' + 4y = 5x^{-2}$, we have:

$\left(2Ax^{-2} - 10Ax^{-2}\ln(x) + 6x^{-2}A(\ln(x))^2\right) + \left(10A\ln(x)\, x^{-2} - 10x^{-2}A(\ln(x))^2\right) + 4A(\ln(x))^2 x^{-2}$

$= 5x^{-2}$

This reduces to: $2Ax^{-2} = 5x^{-2}$

$\Rightarrow 2A = 5 \Rightarrow A = \frac{5}{2}$

$\Rightarrow y_p = A(\ln(x))^2 x^{-2} \Rightarrow y_p = \frac{5}{2}(\ln(x))^2 x^{-2}$

Our general solution is $y = y_p + y_c = \frac{5}{2}(\ln(x))^2 x^{-2} + c_1x^{-2} + c_2\ln(x)\, x^{-2}$

Our general solution is $y = \frac{5}{2}(\ln(x))^2 x^{-2} + c_1x^{-2} + c_2\ln(x)\, x^{-2}$

3. $x^2y'' + 5xy' + 3y = \ln(x)$

First, find the solution to the complementary equation $x^2y'' + 5xy' + 3y = 0$

Our strategy is to seek solutions of the form:

$$y = x^\lambda$$

$$\Rightarrow y' = \lambda x^{\lambda-1}$$

$$\Rightarrow y'' = \lambda(\lambda - 1)x^{\lambda-2} = \left(\lambda^2 - \lambda\right)x^{\lambda-2}$$

Plugging these into the complementary equation $x^2y'' + 5xy' + 3y = 0$, we have:

$$x^2\left(\lambda^2 - \lambda\right)x^{\lambda-2} + 5x\lambda x^{\lambda-1} + 3x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right)x^\lambda + 5\lambda x^\lambda + 3x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right) + 5\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda + 3 = 0$$

$$\Rightarrow (\lambda + 1)(\lambda + 3) = 0$$

$$\Rightarrow \lambda_1 = -1;\ \lambda_2 = -3$$

Our complementary solution is:

$$y_c = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^{-1} + c_2x^{-3}$$

Next, we find our particular solution

Since the right hand side of the equation is **not** a linear combination of powers of x, the Method of Undetermined Coefficients will not work. We must use Variation of Parameters.

Thus, we guess that:

$y = A(x)x^{-1} + B(x)x^{-3}$, where the pair $\{A(x), B(x)\}$ is any pair of functions that make $y = A(x)x^{-1} + B(x)x^{-3}$ the general solution to the original differential equation.

This is the first restriction that we impose on the pair $\{A(x), B(x)\}$.

We still have on restriction left to impose.

We will now compute the derivatives of y and plug them into the original equation:

$x^2y'' + 5xy' + 3y = \ln(x)$.

$$y = A(x)x^{-1} + B(x)x^{-3}$$

$y' = A'(x)\,x^{-1} - A(x)\,x^{-2} + B'(x)\,x^{-3} - 3B(x)\,x^{-4}$

We now impose our second restriction: $A'(x)\,x^{-1} + B'(x)\,x^{-3} = 0$

This results in:

$\Rightarrow y' = -A(x)\,x^{-2} - 3B(x)\,x^{-4}$

$\Rightarrow y'' = -A'(x)\,x^{-2} + 2A(x)\,x^{-3} - 3B'(x)\,x^{-4} + 12B(x)\,x^{-5}$

To find $A(x)$ and $B(x)$ we plug y, y', y'' into the original equation, $x^2y'' + 5xy' + 3y = \ln(x)$.

This yields:

$$\begin{array}{ccccccc} x^2y'' & = & -A'(x) & +2A(x)\,x^{-1} & -3B'(x)\,x^{-2} & +12B(x)\,x^{-3} & \\ +5xy' & = & & -5A(x)\,x^{-1} & & -15B(x)\,x^{-3} & \\ +3y & = & & +3A(x)\,x^{-1} & & +3B(x)\,x^{-3} & \\ \hline x^2y'' + 5xy' + 3y & = & -A'(x) & & -3B'(x)\,x^{-2} & & = \ln(x) \end{array}$$

i.e., $-A'(x) - 3B'(x)\,x^{-2} = \ln(x)$ (Eq. 1)

To eliminate one of the unknown functions, we rely on our second restriction: $A'(x)\,x^{-1} + B'(x)\,x^{-3} = 0$

$\Rightarrow A'(x) + B'(x)\,x^{-2} = 0$

$\Rightarrow A'(x) = -B'(x)\,x^{-2}$ (Eq. 2)

$\Rightarrow -A'(x) = B'(x)\,x^{-2}$

Plugging this into (Eq. 1), we have:

$B'(x)\,x^{-2} - 3B'(x)\,x^{-2} = \ln(x)$

$\Rightarrow -2B'(x)\,x^{-2} = \ln(x)$

$\Rightarrow B'(x) = -\frac{1}{2}x^2\ln(x)$ (Eq. 3)

$$\Rightarrow B(x) = -\frac{1}{2}\int x^2\ln(x)\,dx = -\frac{1}{2}\int \underbrace{\ln(x)}_{u}\underbrace{x^2}_{dv}dx = -\frac{1}{2}\left[uv - \int v\,du\right]$$

$$= -\frac{1}{2}\ln(x)\left(\frac{1}{3}x^3\right) + \frac{1}{2}\int\left(\frac{1}{3}x^3\right)\left(\frac{1}{x}\right)dx$$

$$= -\frac{1}{6}x^3\ln(x) + \frac{1}{6}\int x^2dx = -\frac{1}{6}x^3\ln(x) + \frac{1}{18}x^3 + C_3$$

i.e., $B(x) = -\frac{1}{6}x^3\ln(x) + \frac{1}{18}x^3 + C_3$

To find $A'(x)$, recall, that from (Eq. 3), $B'(x) = -\frac{1}{2}x^2 \ln(x)$

Plugging this into (Eq. 2), we have:

$A'(x) = \frac{1}{2}x^2 \ln(x)\, x^{-2} = \frac{1}{2}\ln(x)$

$\Rightarrow A(x) = \frac{1}{2}\int \ln(x)\, dx = \frac{1}{2}(x\ln(x) - x) + C_4$

i.e., $A(x) = \frac{1}{2}x\ln(x) - \frac{1}{2}x + C_5$

The solution to the original equation is:

$y = A(x)\, x^{-1} + B(x)\, x^{-3}$

$= \left(\frac{1}{2}x\ln(x) - \frac{1}{2}x + C_5\right)x^{-1} + \left(-\frac{1}{6}x^3\ln(x) + \frac{1}{18}x^3 + C_3\right)x^{-3}$

$= \frac{1}{2}\ln(x) - \frac{1}{2} + C_5x^{-1} - \frac{1}{6}\ln(x) + \frac{1}{18} + C_3x^{-3}$

$= \frac{1}{3}\ln(x) - \frac{4}{9} + C_5x^{-1} + C_3x^{-3}$

The solution to the original equation is: $y_g = \frac{1}{3}\ln(x) - \frac{4}{9} + C_5x^{-1} + C_3x^{-3}$