MTH 1125 (2 pm) Test #3 - Solutions

Fall 2022

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 + 6x^2 + 9x + 1$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   1. Compute $f'(x)$ and find critical numbers

      $f'(x) = 3x^2 + 12x + 9$

      a. "Type a" $(f'(c) = 0)$

         Set $f'(x) = 3x^2 + 12x + 9 = 0$

         $\Rightarrow 3x^2 + 12x + 9 = 0$

         $\Rightarrow x^2 + 4x + 3 = 0$

         $\Rightarrow (x+3)(x+1) = 0$

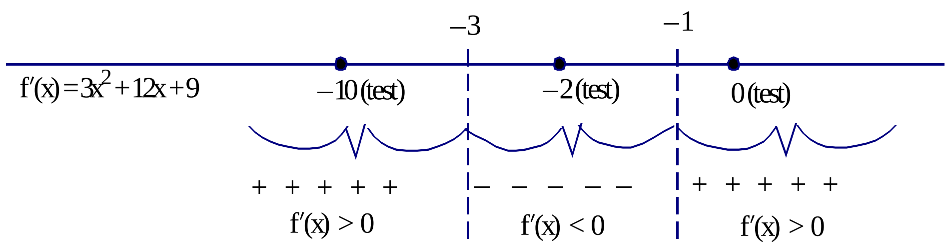
         $\Rightarrow x = -3$; $x = -1$ critical numbers

      b. "Type b" $(f'(c)$ undefined)

         There are none.

   2. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   3. From each interval select a "test point" to plug into $f'(x)$



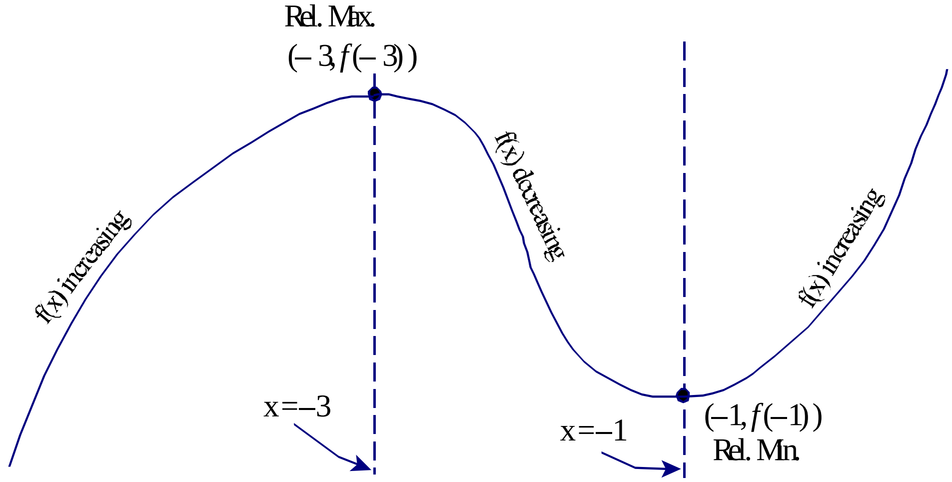
$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(-1, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-3, -1)$

(Because $f'(x)$ is negative on this interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-3, f(-3)) = (-3, 1)$

Rel Min $(-1, f(-1)) = (-1, -3)$

2. $f(x) = \frac{1}{4}x^4 + x^3 + 2x + 4$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection. (Caution - there are **two** points of inflection. Make sure you get them both!)

i. Compute $f''(x)$ and find possible points of inflection

$f'(x) = x^3 + 3x^2 + 2$

$f''(x) = 3x^2 + 6x$

a. "Type a" $(f''(c) = 0)$

Set $f''(x) = 3x^2 + 6x = 0$

$\Rightarrow 3x^2 + 6x = 0$

$\Rightarrow x^2 + 2x = 0$

$\Rightarrow (x + 2)x = 0$

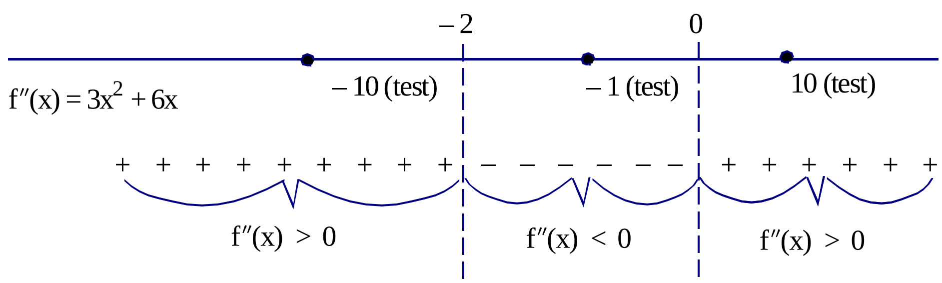
$\Rightarrow x = -2$; and $x = 0$ possible points of inflection

b. "Type b" $(f''(c)$ undefined)

There are none.

ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -2)$ and $(0, \infty)$

(Because $f''(x) > 0$ on these intervals)

$f(x)$ is **concave down** on the interval $(-2, 0)$

(Because $f''(x) < 0$ on this interval)

Since $f(x)$ changes concavity at $x = -2$ and $x = 0$, the points:

$(-2, f(-2)) = (-2, -4)$

and

$(0, f(0)) = (0, 4)$ *are* points of inflection.

3. $f(x) = x^3 + 6x^2 - 36x + 10$ on the interval $[-1, 3]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 3]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 3x^2 + 12x - 36$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 3x^2 + 12x - 36 = 0$

$\Rightarrow 3x^2 + 12x - 36 = 0$

$\Rightarrow x^2 + 4x - 12 = 0$

$\Rightarrow (x + 6)(x - 2) = 0$

$\Rightarrow x = -6;\ x = 2$

Since $-6 \notin [-1, 3]$, we discard it as a critical number

$\Rightarrow x = 2$ "type a" crit. number

b. "Type b" $(f'(c)$ undefined$)$

There are none.

ii. Plug critical numbers and endpoints into the original function.$x^3 + 6x^2 - 36x + 10$

$f(-1) = (-1)^3 + 6(-1)^2 - 36(-1) + 10 = 51 \leftarrow$ Abs Max Value

$f(2) = (2)^3 + 6(2)^2 - 36(2) + 10 = -30 \leftarrow$ Abs Min Value

$f(3) = (3)^3 + 6(3)^2 - 36(3) + 10 = -17$

Abs Max Value $= 51$ (attained at $x = -1$)
Abs Min Value $= -30$ (attained at $x = 2$)

4. $f(x) = \frac{1}{2}x^{\frac{12}{5}} - 12x^{\frac{2}{5}} + 1$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   1. Compute $f'(x)$ and find the critical numbers

      $f'(x) = \frac{6}{5}x^{\frac{7}{5}} - \frac{24}{5}x^{-\frac{3}{5}} = \frac{6x^{\frac{7}{5}}}{5} - \frac{24}{5x^{\frac{3}{5}}} = \frac{6x^{\frac{7}{5}}}{5}\frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} - \frac{24}{5x^{\frac{3}{5}}} = \frac{6x^2-24}{5x^{\frac{3}{5}}}$

      i.e., $f'(x) = \frac{6x^2-24}{5x^{\frac{3}{5}}}$

      a. "Type a" $(f'(c) = 0)$

         Set $f'(x) = 0$ and solve for x

         $\Rightarrow f'(x) = \frac{6x^2-24}{5x^{\frac{3}{5}}} = 0$

         $\Rightarrow 6x^2 - 24 = 0$

         $\Rightarrow x^2 - 4 = 0$

         $\Rightarrow (x+2)(x-2) = 0$

         $\Rightarrow x = -2$ and $x = 2$ are critical numbers.

      b. "Type b" $(f'(c)$ is undefined)

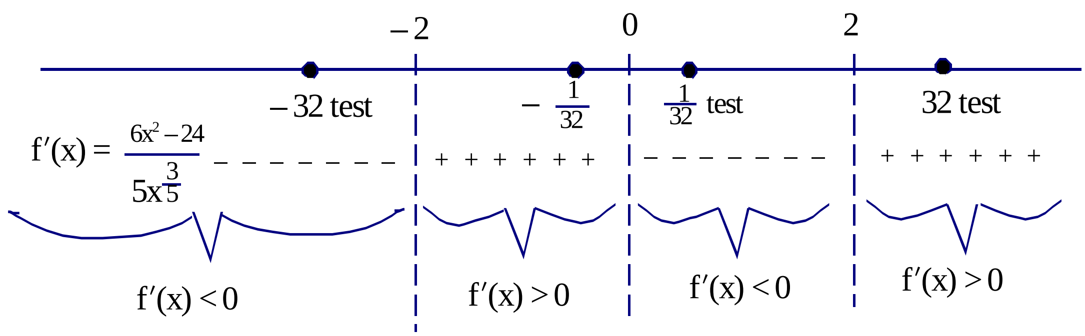
         Look for x-value that causes division by zero.

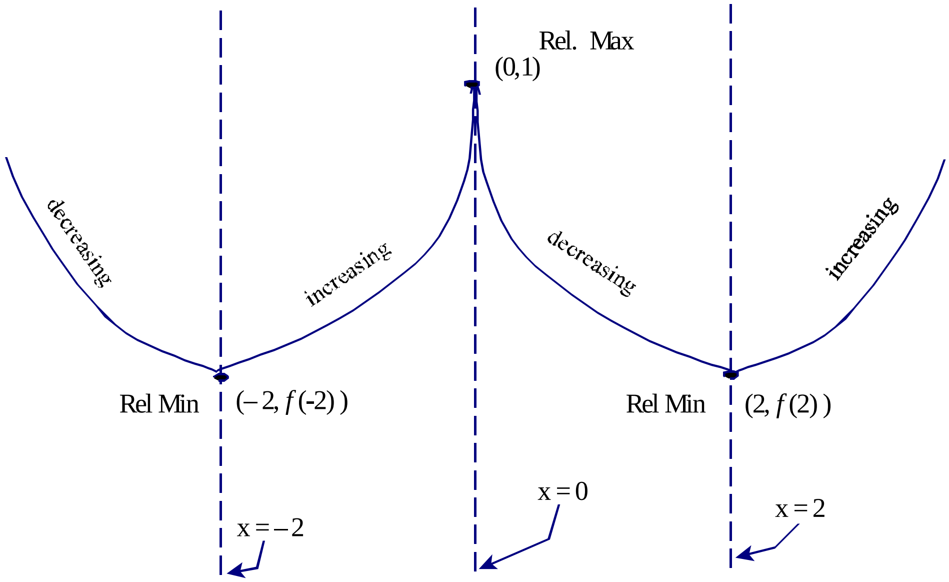
         $\Rightarrow 5x^{\frac{3}{5}} = 0$

         $\Rightarrow x = 0$ "type b" critical number

   2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

   3. Pick a "test point" from each interval to plug into $f'(x)$





$f(x)$ is **increasing** on the interval(s) $(-2,0)$ and $(2,\infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty,-2)$ and $(0,2)$

(because $f'(x)$ is negative on that interval)

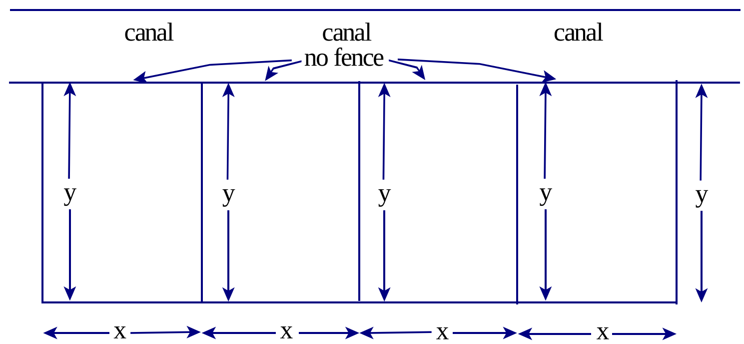
4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

Rel Minimums: $(-2, f(-2))$

and $(2, f(2))$

Rel Maximum: $(0, f(0)) = (0, 1)$

5. Farmer Joe has 1000 yards of fencing with which he will construct a rectangular pen. One side of the pen will border on a straight canal, and no fencing will be required on that side. In addition, Farmer Joe will use some of the fencing to partition the pen into four smaller pens of equal size and similar shape (See picture below). What should the overall dimensions of the pen be in order for the pen to contain the largest area possible?



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = 4xy$

   a. Draw a picture where relevant.

   (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that Joe will use exactly 1000 yards of wire fencing

   Note that Joe will use 1 piece of fence of length $3x$ and four pieces of fence of length y.
   Hence, $4x + 5y = 1000\text{yds}$

   $\Rightarrow 4x = 1000\text{yds} - 5y$

   Plug this into the equation $A = 4xy$, we have:

   $A = (1000\text{yds} - 5y)\, y = 1000\text{yds}\ y - 5y^2$

   $\Rightarrow A(y) = 1000\text{yds}\ y - 5y^2$

3. Determine the restrictions on the independent variable y.

   From the picture, $0\text{yds} \leq y \leq 200\text{yds}$

4. Maximize $A(y)$, using the techniques of Calculus.

   Note that $A(y)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0\text{yds}, 250\text{yds}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers. $A(y) = 1000\text{yds}\ y - 5y^2$

      $A'(y) = 1000\text{yds} - 10y$

      a. "Type a" $(A'(c) = 0)$

         $\Rightarrow A'(y) = 1000\text{yds} - 10y = 0$

         $\Rightarrow 1000\text{yds} - 10y = 0$

         $\Rightarrow -10y = -1000\text{yds}$

         $\Rightarrow y = 100\text{yds}$ critical number

      b. "Type b" ($A'(c)$ is undefined)

         Look for y-values that cause division by zero in $A'(y)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0\text{yds}) = 1000\text{yds}\ (0\text{yds}) - 5(0\text{yds})^2 = 0\text{yds}^2$

      $A(100\text{yds}) = 1000\text{yds}\ (100\text{yds}) - 5(100\text{yds})^2 = 50,000\text{yds}^2 \longleftarrow$ Abs Max Value

      $A(200\text{yds}) = 1000\text{yds}\ (200\text{yds}) - 5(200\text{yds})^2 = 0\text{yds}^2$

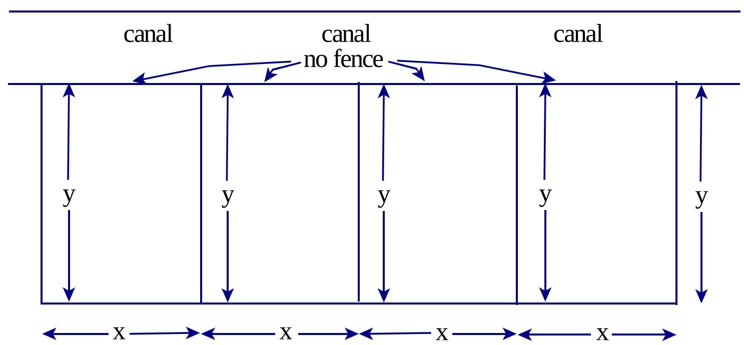
5. Make sure that we've answered the original question.

   1. "What should the overall dimensions of the pen be ..."

      Length $4x = 1000\text{yds} - 5y = 1000\text{yds} - 5(100\text{yds}) = 500\text{yds}$

Length $4x = 500\text{yds}$
Width $y = 100\text{yds}$

Alternative Solution appears on the next page



6. 1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = 4xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that Joe will use exactly 1000 yards of wire fencing

   Note that Joe will use 1 piece of fence of length $3x$ and four pieces of fence of length y.
   Hence, $4x + 5y = 1000\text{yds}$

   $\Rightarrow 5y = 1000\text{yds} - 4x$

   $\Rightarrow y = 200\text{yds} - \frac{4}{5}x$

   Plug this into the equation $A = 4xy$, we have:

   $A = 4x\left(200\text{yds} - \frac{4}{5}x\right) = 800\text{yds } x - \frac{16}{5}x^2$

   $\Rightarrow A(x) = 800\text{yds } x - \frac{16}{5}x^2$

3. Determine the restrictions on the independent variable x.

   From the picture, $0\text{yds} \leq x \leq 250\text{yds}$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0\text{yds}, 250\text{yds}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 800\text{yds} - \frac{32}{5}x$

      a. "Type a" $(A'(c) = 0)$

         $\Rightarrow A'(x) = 800\text{yds} - \frac{32}{5}x = 0$

         $\Rightarrow 800\text{yds} - \frac{32}{5}x = 0$

         $\Rightarrow -\frac{32}{5}x = -800\text{yds}$

         $\Rightarrow x = 125\text{yds}$

         $\Rightarrow x = 125\text{yds}$ critical number

      b. "Type b" $(A'(c)$ is undefined)

         Look for x-values that cause division by zero in $A'(y)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0\text{yds}) = 800\text{yds}\,(0\text{yds}) - \frac{16}{5}(0\text{yds})^2 = 0\text{yds}^2$

      $A(125\text{yds}) = 800\text{yds}\,(125\text{yds}) - \frac{16}{5}(125\text{yds})^2 = 50,000\text{yds}^2 \longleftarrow$ Abs Max Value

      $A(250\text{yds}) = 800\text{yds}\,(250\text{yds}) - \frac{16}{5}(250\text{yds})^2 = 0\text{yds}^2$

5. Make sure that we've answered the original question.

   1. "What should the overall dimensions of the pen be ..."

      Length $4x = 4(125\text{yds}) = 500\text{yds}$

      Width $y = 200\text{yds} - \frac{4}{5}x = 200\text{yds} - \frac{4}{5}(125\text{yds}) = 100\text{yds}$

Length $4x = 500\text{yds}$
Width $y = 100\text{yds}$

EXTRA! (Wow! 10 points!)

In the exercise below,

[1]Deterimine the intervals on which $f(x)$ is increasing/decreasing
[2]Identify all relative maximums and minimums
[3]Deterimine the intervals on which $f(x)$ is CCU/CCD
[4]Identify all points of inflections
[5]Graph $f(x)$

$f(x) = x^3 - 3x^2 - 9x + 13$

(Increasing/Decreasing - Max/Mins)

1. Compute $f'(x)$ and find critical numbers

   $f'(x) = 3x^2 - 6x - 9$

   a. "Type a" ($f'(c) = 0$)

      Set $f'(x) = 3x^2 - 6x - 9 = 0$

      $\Rightarrow 3x^2 - 6x - 9 = 0$

      $\Rightarrow x^2 - 2x - 3 = 0$

      $\Rightarrow (x + 1)(x - 3) = 0$

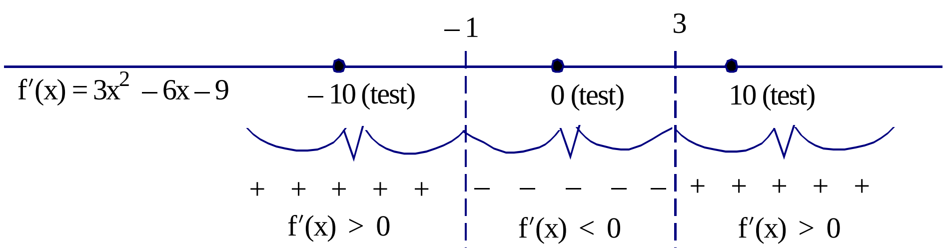
      $\Rightarrow x = -1$; $x = 3$ critical numbers

   b. "Type b" ($f'(c)$ undefined)

      There are none.

2. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

3. From each interval select a "test point" to plug into $f'(x)$



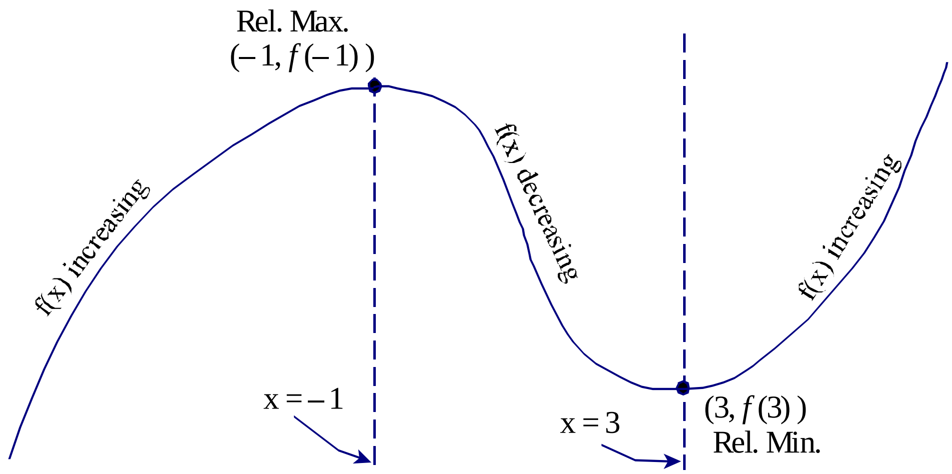
$f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(3, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-1, 3)$

(Because $f'(x)$ is negative on this interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-1, f(-1)) = (-1, 18)$
Rel Min $(3, f(3)) = (3, -14)$

(Concave Up/Concave Down - Points of inflection)

i. Compute $f''(x)$ and find possible points of inflection

$f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

a. "Type a" $(f''(c) = 0)$

Set $f''(x) = 6x - 6 = 0$

$\Rightarrow 6x - 6 = 0$

$\Rightarrow x - 1 = 0$

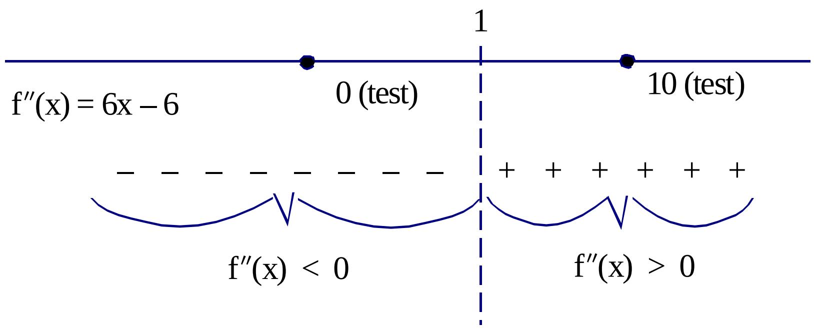
$\Rightarrow x = 1$ possible point of inflection

b. "Type b" ($f''(c)$ undefined)

There are none.

ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave down** on the interval $(-\infty, 1)$

(Because $f''(x) < 0$ on these intervals)

$f(x)$ is **concave up** on the interval $(1, \infty)$

(Because $f''(x) > 0$ on this interval)

Since $f(x)$ changes concavity at $x = 1$, the point:

$(1, f(1)) = (1, 2)$ *is* a point of inflection

Graph of $f(x) = 2x^3 - 12x^2 + 18x - 3$