MTH 3318 Induction Set 1b - Solutions

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Instructions. Prove the following by mathematical induction.

1. Given that $|x_1 + x_2| \leq |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that:

   $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for real numbers $x_1, x_2, \ldots x_n$, (the General Triangle Inequality).

   Proof.

   Step #1: Show that the proposition is true for $n = 1$.

   $|x_1| \leq |x_1|$. True.

   Step #2: Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

   i.e., Assume that $|x_1 + x_2 + x_3 + \ldots + x_k| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k|$ and show that $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

   Observe: $\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \leq |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from Triangle Inequality}}$

   $\leq \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|}_{\text{by Ind. Hyp.}}$.

   i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

   Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all natural numbers, n. ■

2. $(1+x)^n \geq 1+nx$ for any natural number n and any real number $x \geq -1$.

Proof.

Step #1: Show true for $n = 1$

$(1+x)^1 = 1+x \geq 1+(1)\,x$ True.

Step #2: Assume true for $n = k$, and show true for $n = k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number k, and show that

$(1+x)^{k+1} \geq 1+(k+1)\,x$

Observe:

$$(1+x)^{k+1} = \underbrace{(1+x)^k\,(1+x) \geq (1+kx)\,(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)\,x + \underbrace{kx^2}_{kx^2 \geq 0} \geq 1+(k+1)\,x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)\,x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers n and any real number $x \geq -1$ ■

Remark: Our proof hinged on two subtle points:

First, since k is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers x, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

3. For $0 \leq a \leq b$; prove that $a^n \leq b^n$.

Proof.

Step #1: Show true for $n = 1$.

$a^1 = \underbrace{a \leq b}_{\text{given}} = b^1$

i.e., $a^1 \leq b^1$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $a^k \leq b^k$ for some natural number k, and show that $a^{k+1} \leq b^{k+1}$

Observe: $a^{k+1} = \underbrace{a^k \cdot a \leq b^k \cdot a}_{\text{by Ind. Hyp.}} \leq \underbrace{b^k \cdot b}_{a \leq b} = b^{k+1}$

i.e., $a^{k+1} \leq b^{k+1}$

Hence, $a^n \leq b^n$ for all natural numbers, n. ■

4. $n(n+1)$ is divisible by 2 for all natural numbers, n.

Proof.

First, note that a natural number n is divisible by 2 if there exists a natural number m such that $n = 2m$

Step #1: Show true for $n = 1$.

$1((1)+1) = 2 = 2 \cdot 1$

Thus, $n(n+1)$ is divisible by 2, for $n = 1$.

i.e., $1((1)+1) = 2 = 2 \cdot 1$ True.

Step #2: Assume true for $n = k$, and show true for $n = k+1$

i.e., Assume that $k(k+1)$ is divisible by 2, and show that

$(k+1)[(k+1)+1]$ is divisible by 2.

i.e., Assume that $k(k+1) = 2m$, and show that

$(k+1)(k+2)$ is divisible by 2.

Observe: $(k+1)(k+2) = (k+1)k + (k+1)2 = \underbrace{k(k+1) + 2(k+1) = 2m+2}_{\text{by Ind. Hyp.}} =$

$2(m+1)$.

i.e., $(k+1)(k+2) = 2(m+1)$.

i.e., $(k+1)(k+2)$ is divisible by 2.

Hence, $n(n+1)$ is divisible by 2 for all natural numbers, n. ■