MTH 3318 Test #1 - Solutions

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Instructions. Fully document your work.

For problems 1 - 2, prove one using Mathematical Induction:

1. For $0 \leq a \leq b$; prove that $a^n \leq b^n$.

Proof.

Step 1 Show true for $n = 1$.

$$a^1 = \underbrace{a \leq b}_{\text{given}} = b^1$$

i.e., $a^1 \leq b^1$

Step 2 Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $a^k \leq b^k$ for some natural number k, and show that $a^{k+1} \leq b^{k+1}$

Observe: $a^{k+1} = a^k \cdot a = \underbrace{b^k \cdot a}_{\text{by Ind. Hyp.}} = \underbrace{b^k \cdot b}_{a \leq b} = b^{k+1}$

i.e., $a^{k+1} \leq b^{k+1}$

Hence, $a^n \leq b^n$ for all natural numbers, n. ■

2. Given that $\frac{d}{dx}\left[x^0\right] = 0$ and $\frac{d}{dx}\left[x^1\right] = 1$, prove that $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$. You may use the product rule: $\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + g'(x)\,f(x)$.

Proof.

Step 1 Show true for $n = 1$.

$\frac{d}{dx}\left[x^1\right] = 1 = x^0 = x^{1-1}$ True.

Step 2 Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ and show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^{(k+1)-1}$

i.e., show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Observe:

$$\frac{d}{dx}\left[x^{k+1}\right] = \frac{d}{dx}\left[x^k \cdot x\right] = \underbrace{\frac{d}{dx}\left[x^k\right] \cdot x + \frac{d}{dx}\left[x\right] \cdot x^k}_{\text{product rule}} = \underbrace{kx^{k-1}}_{\text{Ind Hyp}} \cdot x + \underbrace{1}_{\text{given}} \cdot x^k$$

$$= kx^k + x^k = (k+1)\,x^k$$

i.e. $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Hence, $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ for all natural numbers n. ■

For problems 3 - 4, prove one using Mathematical Induction:

3. $(1+x)^n \geq 1+nx$ for any natural number n and any real number $x \geq -1$.

Proof.

Step 1 Show true for $n=1$

$$(1+x)^1 = 1+x \geq 1+(1)\,x$$

Step 2 Assume true for $n=k$, and show true for $n=k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number k, and show that $(1+x)^{k+1} \geq 1+(k+1)\,x$

Observe:

$$(1+x)^{k+1} = \underbrace{(1+x)^k(1+x) \geq (1+kx)(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)\,x + \underbrace{kx^2}_{kx^2\geq 0} \geq 1+(k+1)\,x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)\,x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers n and any real number $x \geq -1$ ■

Remark: Our proof hinged on two subtle points:

First, since k is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers x, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

4. Given that $|x_1 + x_2| \leq |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that: $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1|+|x_2|+|x_3|+\ldots+|x_n|$ (the General Triangle Inequality).

Proof.

Step 1 Show true for $n = 1$

$|x_1| \leq |x_1|$

Step 2 Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that:

$|x_1 + x_2 + x_3 + \ldots + x_k| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k|$

and show that:

$|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Observe:

$$\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \leq |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from given}}$$

$$\leq \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k|}_{\text{by Ind. Hyp.}} + |x_{k+1}|$$

i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all natural numbers, n. ■

For problems 5- 9 prove three using Mathematical Induction.

5. $1+3+5+\ldots+(2n-1)=n^2$

   i.e. $\sum_{i=1}^{n}(2i-1)=n^2$

   Proof.

Step 1 Show true for $n=1$

$\sum_{i=1}^{1}(2i-1)=(2(1)-1)=1=(1)^2$

Step 2 Assume true for $n=k$, and show true for $n=k+1$

i.e., Assume that: $\sum_{i=1}^{k}(2i-1)=k^2$ for some natural number k,

and show that:

$\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$

Observe:

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= \underbrace{\sum_{i=1}^{k}(2i-1)+(2(k+1)-1)=k^2+(2(k+1)-1)}_{\text{by Induction Hypothesis}} \\ &= k^2+2k+1=(k+1)^2\end{aligned}$$

i.e., $\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$

Hence, $\sum_{i=1}^{n}(2i-1)=n^2$ for all natural numbers, n. ■

6. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

i.e. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof.

Step 1 Show that the proposition is true for $n = 1$.

$\sum_{i=1}^{1} i = 1 = \frac{(1)((1)+1)}{2}$ True.

Step 2 **Step #2:** Assume that the proposition is true for $n = k,$ and prove that the proposition is true for $n = k + 1$.

i.e., Assume that: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true for some natural number k,

and prove that:

$\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$ is true.

(Equivalently, prove that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.)

Observe: $\sum_{i=1}^{k+1} i = \underbrace{\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)}_{\text{by Induction Hypothesis}} = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

i.e., $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Hence, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all natural numbers, n. ■

7. $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

i.e. $\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$

Proof.

Step 1 Show that the proposition is true for $n = 1$

$\sum_{j=1}^{1} \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{(1)(3)} = \frac{1}{3} = \frac{1}{2(1)+1}$

Step 2 Assume true for $n = k,$ and show true for $n = k + 1$

i.e., Assume that $\sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} = \frac{k}{2k+1}$ for some natural number k, and show

that $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2(k+1)+1}$

i.e., $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2k+3}$

Observe:

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} &= \sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \text{ (by Induction Hypothesis)} \\
&= \frac{k}{2k+1} \cdot \frac{2k+3}{2k+3} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{(2k+3)}
\end{aligned}$$

i.e., $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2k+3}$

Hence, $\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$ for all natural numbers, n. ■

8. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

i.e. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Proof.

Step 1 Show true for $n = 1$

$\sum_{i=1}^{1} i^3 = (1)^3 = 1 = \frac{(1)^2((1)+1)^2}{4}$

Step 2 Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that:

$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ for some natural number k,

and show that:

$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

i.e., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$

Observe:

$$\sum_{i=1}^{k+1} i^3 = \underbrace{\sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right] = \frac{(k+1)^2(k+2)^2}{4}$$

i.e., $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers, n. ■

9. $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 < \frac{n^4}{4}$ all natural numbers, n.

i.e., $\sum_{i=1}^{n} (i-1)^3 < \frac{n^4}{4}$

Proof.

Step 1 Show true for $n = 1$

i.e., show that: $\sum_{i=1}^{1} (i-1)^3 < \frac{1^4}{4}$

$\sum_{i=1}^{1} (i-1)^3 = (1-1)^3 = 0 < \frac{1}{4} = \frac{1^4}{4}$

i.e., $\sum_{i=1}^{1} (i-1)^3 < \frac{1^4}{4}$

Step 2 Assume true for $n = k$, and show true for $n = k+1$

i.e., Assume that: $\sum_{i=1}^{k} (i-1)^3 < \frac{k^4}{4}$ for some natural number k,

and show that:

$\sum_{i=1}^{k+1} (i-1)^3 < \frac{(k+1)^4}{4}$

Observe:

$$\sum_{i=1}^{k+1} (i-1)^3 = \underbrace{\sum_{i=1}^{k} (i-1)^3 + ((k+1)-1)^3 < \frac{k^4}{4} + k^3}_{\text{by Induction Hypothesis}} = \frac{k^4}{4} + \frac{4k^3}{4} = \frac{k^4+4k^3}{4}$$

$$< \frac{k^4+4k^3+6k^2+4k+1}{4} = \frac{(k+1)^4}{4}$$

i.e., $\sum_{i=1}^{k+1} (i-1)^3 < \frac{(k+1)^4}{4}$

Hence, $\sum_{i=1}^{n} (i-1)^3 < \frac{n^4}{4}$ for all natural numbers, n. ■

10. $\frac{n^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + n^3$ all natural numbers, n.

i.e., $\frac{n^4}{4} < \sum_{i=1}^{n} i^3$

Proof.

Step 1 Show true for $n = 1$

$\frac{1^4}{4} < 1^3 = \sum_{i=1}^{1} i^3$

i.e., $\frac{n^4}{4} < \sum_{i=1}^{n} i^3$

Step 2 Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that: $\frac{k^4}{4} < \sum_{i=1}^{k} i^3$ for some natural number k,

and show that:

$\frac{(k+1)^4}{4} < \sum_{i=1}^{k+1} i^3$ (Or Equivalently: $\sum_{i=1}^{k+1} i^3 > \frac{(k+1)^4}{4}$)

Observe:

$$\sum_{i=1}^{k+1} i^3 = \underbrace{\sum_{i=1}^{k} i^3 + (k+1)^3 > \frac{k^4}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^4}{4} + \frac{4(k+1)^3}{4} = \frac{k^4+4k^3+12k^2+12k+4}{4}$$

$$> \frac{k^4+4k^3+6k^2+4k+1}{4} = \frac{(k+1)^4}{4}$$

i.e., $\sum_{i=1}^{k+1} i^3 > \frac{(k+1)^4}{4}$ (Or Equivalently: $\frac{(k+1)^4}{4} < \sum_{i=1}^{k+1} i^3$)

Hence, $\frac{n^4}{4} < \sum_{i=1}^{n} i^3$ for all natural numbers, n. ■