MTH 4441 - HW 5 - Isomorphisms - Solutions

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In Exercises 1-12, determine whether the two groups are isomorphic. If they aren't, give at least one reason why. If they are, justify your answer either by exhibiting an isomorphism between the two groups, or by proving that they are isomorphic by some other method.

1. $(\mathbb{Z},+)$ and $(2\mathbb{Z},+)$

   ARE isomorphic.

   $f:\mathbb{Z}\rightarrow 2\mathbb{Z}$, given by $f(n)=2n$ is one to one and onto.

   $f(n_1+n_2)=2(n_1+n_2)=2n_1+2n_2=f(n_1)+f(n_2)$

   i.e., $f(n_1+n_2)=f(n_1)+f(n_2)$

2. $(2\mathbb{Z},+)$ and $(6\mathbb{Z},+)$

   ARE isomorphic.

   $f:2\mathbb{Z}\rightarrow 6\mathbb{Z}$, given by $f(n)=3n$ is one to one and onto.

   $f(n_1+n_2)=3(n_1+n_2)=3n_1+3n_2=f(n_1)+f(n_2)$

   i.e., $f(n_1+n_2)=f(n_1)+f(n_2)$

   Alternatively:

   $f:2\mathbb{Z}\rightarrow 6\mathbb{Z}$, given by $f(2n)=6n$ is one to one and onto.

   $f(2n_1+2n_2)=f(2(n_1+n_2))=6(n_1+n_2)=6n_1+6n_2=f(2n_1)+f(2n_2)$

   i.e., $f(2n_1+2n_2)=f(2n_1)+f(2n_2)$

3. $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$ and the group $(G, *)$ whose group table is given below:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

ARE isomorphic.

Perhaps the best way to show that these two groups are isomorphic is to show that the group table for one group can be obtained from the group table of the other by renaming the elements.

Rename the elements of $(G, *)$ as follows:

e as $(0, 0)$

a as $(1, 0)$

b as $(0, 1)$

c as $(1, 1)$

And the group table for $(G, *)$:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

becomes

$\oplus$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

which is the group table for $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$.

4. $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$ and the group $(G, *)$ whose group table is given below:

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Are NOT isomorphic.

Comparing the group table for $(G, *)$ with the group table for $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$ (below), we see that:

1) Every element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is its own inverse, while e and b are the only elements of $(G, *)$ having this property.

2) $(G, *)$ is cyclic, with generators a and c, while $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$ is not cyclic.

$\oplus$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

5. The groups $(G, *)$ and $(H, *)$, whose group tables are given below:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Are NOT isomorphic.

Comparing the group table for $(G, *)$ with the group table for $(H, *)$, we see that:

1) Every element of $(G, *)$ is its own inverse, while e and b are the only elements of $(H, *)$ having this property.

2) $(H, *)$ is cyclic, with generators a and c, while $(G, *)$ is not cyclic.

6. The groups $(G,*)$ and $(H,*)$, whose group tables are given below:

ARE isomorphic.

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$*$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

$(G,*)$ is cyclic, of order 4, with generators a and c.

$(H,*)$ is cyclic, of order 4, with generators a and b.

Since cyclic groups of the same order are isomorphic, $(G,*)$ and $(H,*)$ are isomorphic.

7. The groups $(G,*)$ and $(H,*)$, whose group tables are given below:

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$*$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	e

Are NOT isomorphic. Both groups are finite with a different number of elements. Therefore, there cannot be a one to one and onto function between the two groups.

8. The groups $(\mathbb{Z}_4,\oplus)$ and $(G,*)$, whose group table is given below:

$(G,*)$:

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$(\mathbb{Z}_4,\oplus)$:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

ARE isomorphic.

$(G,*)$ is cyclic, of order 4, with generators a and c.

$(\mathbb{Z}_4,\oplus)$ is cyclic, of order 4, with generators 1 and 3.

Since cyclic groups of the same order are isomorphic, $(G,*)$ and $(\mathbb{Z}_4,\oplus)$ are isomorphic.

Alternatively: The group table for $(\mathbb{Z}_4,\oplus)$ can be obtained from the group table of $(G,*)$, by renaming the elements e,a,b,c as $0,1,2,3$, respectively.

9. The groups $(\mathbb{Z}_6, \oplus)$ and $(H, *)$, whose group tables are given below:

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$*$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e

Are NOT isomorphic.

Looking at the groups tables:

We see that $(\mathbb{Z}_6, \oplus)$ is abelian. Looking at the group table of $(H, *)$, we see that $(H, *)$ is NOT abelian. For example, $c * a = f$ and $a * c = d$.

We see that $(\mathbb{Z}_6, \oplus)$ is cyclic, with generators $1, 3, 5$, while no element of $(H, *)$ has order greater than three.

10. The groups $(\mathbb{Z}_8, \oplus)$ and $(H, *)$, whose group tables are given below:

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

$*$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Are NOT isomorphic.

Looking at the groups tables:

We see that $(\mathbb{Z}_8, \oplus)$ is abelian. Looking at the group table of $(H, *)$, we see that $(H, *)$ is NOT abelian. For example, $i * j = k$ and $j * i = -k$.

We see that $(\mathbb{Z}_8, \oplus)$ is cyclic, with generators $1, 3, 5, 7$, while no element of $(H, *)$ has order greater than four.

11. $(\mathbb{Z}_2 \times \mathbb{Z}_3, \oplus)$ and $(\mathbb{Z}_3 \times \mathbb{Z}_2, \oplus)$

    ARE isomorphic.

    Note that both groups are cyclic of order 6. Hence, isomorphic.

12. $(\mathbb{Z}_3 \times \mathbb{Z}_2, \oplus)$ and $(\mathbb{Z}_6, \oplus)$

    ARE isomorphic.

    Both groups are cyclic of order 6. Hence, isomorphic.

In Exercises 13-17, determine whether the given function is an isomorphism between the two groups. If it is, show that it satisfies all of the properties of an isomorphism. If it is not, give at least one reason why the function is NOT an isomorphism.

13. $f : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{R}^+, \cdot)$, given by $f(x) = |x|$

    NOT an isomorphism.

    Oberve: $f(1) = 1$ and $f(-1) = 1$

    i.e., $f(1) = f(-1)$

    Therefore, f is not one to one

14. $f : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{Z}_2 \times \mathbb{R}^+, *)$, given by $f(x) = \begin{cases} (0, x) & \text{if } x > 0 \\ (1, -x) & \text{if } x < 0 \end{cases}$,

where, $*$ in $(\mathbb{Z}_2 \times \mathbb{R}^+, *)$ is addition modulo 2 in the first coordinate, and multiplication in the second coordinate.

IS an isomorphism.

To see that f is one to one and onto, note that f maps $\mathbb{R}^+$ to $(0, \mathbb{R}^+)$ and $\mathbb{R}^-$ to $(1, \mathbb{R}^+)$, in the following manner.

For $x \in \mathbb{R}^+$, $f(x) = (0, x)$

For $x \in \mathbb{R}^-$, $f(x) = (1, -x)$

Both are clearly one to one and onto.

Thus, $f : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{Z}_2 \times \mathbb{R}^+, *)$ is clearly one to one and onto.

Next, we must show that $f(x \cdot y) = f(x) * f(y)$

Observe:

For $x, y \in \mathbb{R}^+$, $f(x \cdot y) = (0, x \cdot y) = (0 \oplus 0, x \cdot y) = (0, x) * (0, y) = f(x) * f(y)$

For $x, y \in \mathbb{R}^-$, $f(x \cdot y) = (0, x \cdot y) = (1 \oplus 1, x \cdot y) = (1, x) * (1, y) = f(x) * f(y)$

For $x \in \mathbb{R}^+, y \in \mathbb{R}^-$, $f(x \cdot y) = (1, x \cdot y) = (0 \oplus 1, x \cdot y) = (0, x) * (1, y) = f(x) * f(y)$

For $x \in \mathbb{R}^-, y \in \mathbb{R}^+$, $f(x \cdot y) = (1, x \cdot y) = (1 \oplus 0, x \cdot y) = (1, x) * (0, y) = f(x) * f(y)$

In all cases, $f(x \cdot y) = f(x) * f(y)$

Thus, f is an isomorphism.

15. $f : (\mathbb{R}^+, \cdot) \to (\mathbb{R}^+, \cdot)$, given by $f(x) = \sqrt{x}$

Oddly enough, f does turn out to be one to one and onto. But before we go to the trouble of establishing this, let's determine whether or not $f(x \cdot y) = f(x) \cdot f(y)$

Observe: $f(x \cdot y) = \sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y} = f(x) \cdot f(y)$

OK, let's show that f is one to one and onto.

Suppose that $f(x_1) = f(x_2)$

$\Rightarrow \sqrt{x_1} = \sqrt{x_2}$

$\Rightarrow \left(\sqrt{x_1}\right)^2 = \left(\sqrt{x_2}\right)^2$

$\Rightarrow x_1 = x_2$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence, f is one to one.

To show that f is onto, let $y \in \mathbb{R}^+$

Observe that $f(y^2) = \sqrt{y^2} = y$

Thus, given $y \in \mathbb{R}^+$, $\exists\ x \in \mathbb{R}^+$ (namely $x = y^2$) such that $f(x) = y$

Hence, f is one to one and onto.

Therefore, f is an isomorphism.

16. $f : (\mathbb{Z}_2 \times \mathbb{Z}_3, \oplus)$ and $(\mathbb{Z}_3 \times \mathbb{Z}_2, \oplus)$ given by $f((a,b)) = (b,a)$

IS an isomorphism.

To see that f is **one to one,** note that:

$f((a_1, b_1)) = f((a_2, b_2))$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

i.e., $f((a_1, b_1)) = f((a_2, b_2)) \Rightarrow (a_1, b_1) = (a_2, b_2)$

Hence, f is one to one.

To see that f is **one to one,** note that:

Given $(b,a) \in \mathbb{Z}_3 \times \mathbb{Z}_2$, $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ is such that $f((a,b)) = (b,a)$

Hence, f is one to one.

Next, note that $f((a_1, b_1) \oplus (a_2, b_2)) = f((a_1 \oplus a_2, b_1 \oplus b_2)) = (b_1 \oplus b_2, a_1 \oplus a_2)$

$$= (b_1, a_1) \oplus (b_2, a_2) = f(a_1, b_1) \oplus f(a_2, b_2)$$

i.e., $f((a_1, b_1) \oplus (a_2, b_2)) = f(a_1, b_1) \oplus f(a_2, b_2)$

Hence, $f : (\mathbb{Z}_2 \times \mathbb{Z}_3, \oplus) \to (\mathbb{Z}_3 \times \mathbb{Z}_2, \oplus)$ given by $f((a,b)) = (b,a)$ is an isomorphism.

17. $f : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$, given by $f(x) = \ln(x)$

Note that $g : (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ is such that $(g \circ f) : (\mathbb{R}^+, \cdot) \to (\mathbb{R}^+, \cdot)$ is the identity on $(\mathbb{R}^+, \cdot)$.

Hence, $f : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$ is one to one and onto.

Next observe: $f(x_1 \cdot x_2) = \ln(x_1 \cdot x_2) = \ln(x_1) + \ln(x_2) = f(x_1) + f(x_2)$

Thus, f is an isomorphism.