MTH 4441 Homework Exercises #1

Fall 2024

Pat Rossi Name ____________________

1. In each case below, determine whether $*$ is a **closed** binary operation on the given set. If it *IS* a **closed** binary operation, then determine whether it is commutative and/or associative.

   (a) $(\mathbb{Z}, *)$ where $a * b = a + b^2$

   IS a closed binary operation on $\mathbb{Z}$

   NOT commutative. Given $a \neq b$, $a * b = a + b^2 \neq b + a^2 = b * a$

   For example: $1 * 2 = 1 + 2^2 = 5 \neq 3 = 2 + 1^2 = 2 * 1$

   i.e., $1 * 2 \neq 2 * 1$

   NOT associative.

   $(a * b) * c = (a + b^2) * c = (a + b^2) + c^2 = a + b^2 + c^2$

   $a * (b * c) = a * (b + c^2) = a + (b + c^2)^2 = a + b^2 + 2bc^2 + c^4$

   $(a * b) * c \neq a * (b * c) = a + b^2 + c^2$

   For example: $(1 * 2) * 3 = 1 + 2^2 + 3^2 = 14$

   Whereas: $1 * (2 * 3) = 1 + 2^2 + 2\,(2)\,(3)^2 + 3^4 = 122$

   i.e., $(1 * 2) * 3 \neq 1 * (2 * 3)$

   (b) $(\mathbb{Z}, *)$ where $a * b = a^2 b^3$

   IS a closed binary operation on $\mathbb{Z}$

   NOT commutative. Given $a \neq b$, $a * b = a^2 b^3 \neq b^2 a^3 = b * a$

   NOT associative.

   $(a * b) * c = a^2 b^3 * c = (a^2 b^3)^2 c^3 = a^4 b^6 c^3$

   $a * (b * c) = a * b^2 c^3 = a^2 (b^2 c^3)^3 = a^2 b^6 c^9$

   $(a * b) * c \neq a * (b * c)$

   (c) $(\mathbb{R}, *)$ where $a * b = \frac{a}{a^2 + b^2}$

   NOT a closed binary operation on $\mathbb{Z}$ (e.g., $0 * 0$ is undefined, and hence, $0 * 0$ is not assigned an element of $\mathbb{R}$)

(d) $(\mathbb{Z}, *)$ where $a * b = \frac{a^2+2ab+b^2}{a+b}$

NOT a closed binary operation on $\mathbb{Z}$ (e.g., $0 * 0$ is undefined, and hence, $0 * 0$ is not assigned an element of $\mathbb{Z}$)

(e) $(\mathbb{Z}, *)$ where $a * b = a + b - ab$

IS a closed binary operation on $\mathbb{Z}$

IS Commutative: $a * b = a + b - ab = b + a - ba = b * a$

i.e., $a * b = b * a$

IS Associative:

$$(a * b) * c = (a + b - ab) * c = (a + b - ab) + c - (a + b - ab)\, c$$

$$= (a + b - ab) + c - ac - bc + abc = a + b + c - ab - ac - bc + abc$$

$$a * (b * c) = a * (b + c - bc) = a + (b + c - bc) - a\,(b + c - bc)$$

$$= a + b + c - bc - ab - ac + abc = a + b + c - ab - ac - bc + abc$$

i.e., $(a * b) * c = a * (b * c)$

(f) $(\mathbb{R}, *)$ where $a * b = b$

IS a closed binary operation on $\mathbb{Z}$

Is NOT Commutative: Given $a \neq b$, $a * b = b \neq a = b * a$

i.e., $a * b \neq b * a$

IS Associative:

$(a * b) * c = b * c = c$

$a * (b * c) = a * c = c$

i.e., $(a * b) * c = a * (b * c)$

(g) $(S, *)$ where $S = \{-4, -2, 1, 2, 3\}$, and $a * b = |b|$

Is NOT a closed binary operation on S. (e.g., $1 * (-4) = |4| = 4$, and hence, $1 * (-4)$ is not assigned an element of S)

(h) $(\{1, 2, 3, 6, 18\}, *)$ where $a * b = ab$

NOT a closed binary operation on $\mathbb{Z}$ ($6 * 6 = 36 \notin \{1, 2, 3, 6, 18\}$. Hence. the operation is not closed on the set)

(i) $\left(\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}\right\}, *\right)$ where $*$ is matrix addition

IS a closed binary operation on $\mathbb{Z}$

IS Commutative: Given $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, we have:

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} = \begin{bmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{bmatrix} = B + A$$

i.e., $A + B = B + A$

IS Associative: Given $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$; $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$, we have:

$$(A + B) + C = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 & (a_2 + b_2) + c_2 \\ (a_3 + b_3) + c_3 & (a_4 + b_4) + c_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + (b_1 + c_1) & a_2 + (b_2 + c_2) \\ a_3 + (b_3 + c_3) & a_4 + (b_4 + c_4) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 + c_1 & b_2 + c_2 \\ b_3 + c_3 & b_4 + c_4 \end{bmatrix}$$

$= A + (B + C)$

i.e., $(A + B) + C = A + (B + C)$