Problem Set 2.4; page 31

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1. Find the $\gcd(143, 227)$, $\gcd(306, 657)$, $\gcd(272, 1479)$.

   We find all of these by using the Division Algorithm.

   (a) $\gcd(143, 227)$

   $277 = q_1(143) + r_1$

   $277 = (1)(143) + 134$

   Repeat with 143 and 134.

   $143 = q_2(134) + r_2$

   $143 = (1)(134) + 9$

   Repeat with 134 and 9.

   $134 = q_3(9) + r_3$

   $134 = (14)(9) + 8$

   Repeat with 9 and 8.

   $9 = q_4(8) + r_4$

   $9 = (1)(8) + 1$

   Repeat with 8 and 1.

   $8 = q_5(1) + r_1$

   $8 = (8)(1) + 0$

   $\gcd(143, 227) =$ last non-zero remainder

   $\gcd(143, 227) = 1$

(b) $\gcd(306, 657)$

$657 = q_1(306) + r_1$

$657 = (2)(306) + 45$

Repeat with 306 and 45

$306 = q_2(45) + r_2$

$306 = (6)(45) + 36$

Repeat with 45 and 36

$45 = q_3(36) + r_3$

$45 = (1)(36) + 9$

Repeat with 36 and 9.

$36 = q_4(9) + r_4$

$36 = (4)(9) + 0$

$\gcd(306, 657) =$ the last non-zero remainder.

$\gcd(306, 657) = 9$

(c) $\gcd(272, 1479)$

$1479 = q_1 (272) + r_1$

$1479 = (5)(272) + 119$

Repeat with 272 and 119.

$272 = q_2 (119) + r_1$

$272 = (2)(119) + 34$

Repeat with 119 and 34.

$119 = q_3 (34) + r_3$

$119 = (3)(34) + 17$

Repeat with 34 and 17.

$34 = q_4 (17) + r_4$

$34 = (2)(17) + 0$

$\gcd(272, 1479) = \text{last non-zero remainder}$

$\gcd(272, 1479) = 17$

2. Use the Euclidean Algorithm to obtain integers x and y satisfying the following:

(a) $\gcd(56, 72) = 56x + 72y$

We find $\gcd(56, 72)$ by using the Euclidean Algorithm, and then "retracing our steps."

$72 = q_1\,(56) + r_1$

$72 = (1)\,(56) + 16$ (eq. 2)

Repeat using 56 and 16

$56 = q_2\,(16) + r_2$

$56 = (3)\,(16) + 8$ (eq. 1)

Repeat using 16 and 8.

$16 = q_3\,(8) + r_3$

$16 = (2)\,(8) + 0$

$\gcd(56, 72) =$ last non-zero remainder

$\gcd(56, 72) = 8$

So, we want x and y such that $56x + 72y = 8$

$8 = 56 - (3)\,(16)$ (From eq. 1)

$16 = 72 - (1)\,(56)$ (From eq.2)

$\Rightarrow 8 = 56 - (3)\,(72 - (1)\,(56))$

$\Rightarrow 8 = (4)\,(56) - (3)\,(72)$

i.e., $56\,(4) + 72\,(-3) = 8$

Our particular solution is $(x_0, y_0) = (4, -3)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$; for $t \in \mathbb{Z}$

$$= \left(\tfrac{72}{8}t, -\tfrac{56}{8}t\right) = (9t, -7t)\,; \text{ for } t \in \mathbb{Z}$$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (4, -3) + (9t, -7t) = (4 + 9t, -3 - 7t)\,; \quad \text{for } t \in \mathbf{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (4 + 9t, -3 - 7t)\,; \quad \text{for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 4 + 9t; \qquad y = -3 - 7t \qquad \text{for } t \in \mathbf{Z}$$

(b) $\gcd(24, 138) = 24x + 138y$

We find $\gcd(24, 138)$ by using the Euclidean Algorithm, and then "retracing our steps."

$138 = q_1(24) + r_1$

$138 = (5)(24) + 18$ (eq. 2)

Repeat using 24 and 18.

$24 = q_2(18) + r_2$

$24 = (1)(18) + 6$ (eq. 1)

Repeat using 18 and 6.

$18 = q_3(6) + r_3$

$18 = (3)(6) + 0$

$\gcd(24, 138) = \text{last non-zero remainder}$

$\gcd(24, 138) = 6$

So, we want x and y such that $24x + 138y = 6$

$6 = 24 - (1)(18)$ (From eq. 1)

$18 = 138 - (5)(24)$ (From eq. 2)

$\Rightarrow 6 = 24 - (1)(138 - (5)(24))$

$\Rightarrow 6 = (6)(24) - (1)(138)$

i.e., $24(6) + 138(-1) = 6$

Our particular solution is $(x_0, y_0) = (6, -1)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$; for $t \in \mathbb{Z}$

$$= \left(\tfrac{138}{6}t, -\tfrac{24}{6}t\right) = (23t, -4t); \text{ for } t \in \mathbb{Z}$$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (6, -1) + (23t, -4t) = (6 + 23t, -1 - 4t); \text{ for } t \in \mathbf{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (6 + 23t, -1 - 4t); \text{ for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 6 + 23t; \quad y = -1 - 4t \quad \text{for } t \in \mathbf{Z}$$

(c) $\gcd(119, 272) = 119x + 272y$

We find $\gcd(119, 272)$ by using the Euclidean Algorithm, and then "retracing our steps."

$272 = q_1 119 + r_1$

$272 = (2)(119) + 34$ (eq. 2)

Repeat for 119 and 34

$119 = q_2(34) + r_2$

$119 = (3)(34) + 17$ (eq. 1)

Repeat for 34 and 17

$34 = q_3(17) + r_3$

$34 = (2)(17) + 0$

$\gcd(119, 272) =$ last non-zero remainder

$\gcd(119, 272) = 17$

So, we want x and y such that $119x + 272y = 17$

$17 = 119 - (3)(34)$ (From eq. 1)

$34 = 272 - (2)(119)$(From eq. 2)

$\Rightarrow 17 = 119 - (3)(272 - (2)(119))$

$\Rightarrow 17 = (7)(119) - (3)(272)$

i.e., $119(7) + 272(-3) = 17$

Our particular solution is $(x_0, y_0) = (7, -3)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$; for $t \in \mathbb{Z}$

$= \left(\frac{272}{17}t, -\frac{119}{17}t\right) = (16t, -7t)$; for $t \in \mathbb{Z}$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (7, -3) + (16t, -7t) = (7 + 16t, -3 - 7t); \quad \text{for } t \in \mathbf{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (7 + 16t, -3 - 7t); \quad \text{for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 7 + 16t; \qquad y = -3 - 7t \qquad \text{for } t \in \mathbf{Z}$$

(d) $\gcd(1769, 2378) = 1769x + 2378y$

Find $\gcd(1769, 2378)$ by using the Euclidean Algorithm and retracing steps.

$2378 = q_1(1769) + r_1$

$2378 = (1)(1769) + 609$ (eq. 4)

Repeat for 1769 and 609

$1769 = q_2(609) + r_2$

$1769 = (2)(609) + 551$ (eq. 3)

Repeat with 609 and 551

$609 = q_3(551) + r_3$

$609 = (1)(551) + 58$ (eq. 2)

Repeat with 551 and 58

$551 = q_4(58) + r_4$

$551 = (9)(58) + 29$ (eq. 1)

Repeat with 58 and 29

$58 = q_5(29) + r_5$

$58 = (2)(29) + 0$

$\gcd(1769, 2378) =$ last non-zero remainder

$\gcd(1769, 2378) = 29$

So, we want x and y such that $1769x + 2378y = 29$

$29 = 551 - (9)(58)$ (From eq. 1)

$58 = 609 - (1)(551)$ (From eq. 2)

$\Rightarrow 29 = 551 - (9)(609 - (1)(551))$

$\Rightarrow 29 = (-9)(609) + (10)(551)$

$551 = 1769 - (2)(609)$ (From eq. 3)

$\Rightarrow 29 = (-9)(609) + (10)(1769 - (2)(609))$

$\Rightarrow 29 = (10)(1769) + (-29)(609)$

$609 = 2378 - (1)(1769)$ (From eq. 4)

$\Rightarrow 29 = (10)(1769) + (-29)(2378 - (1)(1769))$

$\Rightarrow 29 = (39)(1769) + (-29)(2378)$

i.e., $1769(39) + 2378(-29) = 29$

Our particular solution is $(x_0, y_0) = (39, -29)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$; for $t \in \mathbb{Z}$

$$= \left(\tfrac{2378}{29}t, -\tfrac{1769}{29}t\right) = (82t, -61t)\,; \quad \text{for } t \in \mathbb{Z}$$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (39, -29) + (82t, -61t) = (39 + 82t, -29 - 61t)\,; \quad \text{for } t \in \mathbf{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (39 + 82t, -29 - 61t)\,; \quad \text{for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 39 + 82t; \qquad y = -29 - 61t \qquad \text{for } t \in \mathbf{Z}$$

4. (a) If $\gcd(a, b) = 1$, prove that $\gcd(a+b, a-b) = 1$ or 2.

Proof. Suppose that $\gcd(a, b) = 1$.

Let $d = \gcd(a+b, a-b)$.

This means that d is a *common* divisor of $a+b$ and $a-b$, and hence, of their sum, $(a+b)+(a-b) = 2a$.

Similarly, d divides the difference of $a+b$ and $a-b$. (i.e., d divides $(a+b)-(a-b) = 2b$.)

Since d is a *common* divisor of $2a$ and $2b$, it follows that

$d \le \gcd(2a, 2b) = 2\gcd(a, b) = 2 \cdot 1 = 2.$

i.e., $d \le 2$.

Hence, $d = \gcd(a+b, a-b) = 1$ or 2. ■

4. (b) $\gcd(2a+b, a+2b) = 1$ or 3.

Proof.

Let $d = \gcd(2a+b, a+2b)$.

This means that d is a *common* divisor of $2a+b$ and $a+2b$, and hence, d divides any linear combination of $2a+b$ and $a+2b$.

In particular, d divides $2(2a+b)-(a+2b) = 3a$.

Also, d divides $-(2a+b)+2(a+2b) = 3b$.

i.e., d is a *common* divisor of $3a$ and $3b$.

$\Rightarrow d \mid \gcd(3a, 3b)$

Since $\underbrace{\gcd(3a, 3b) = 3 \cdot \gcd(a, b)}_{\text{By Thm 2.7}} = 3$, this implies that $d \mid 3$

Hence, $d = 1$ or $d = 3$ ■