MTH 3318 Test #1 - Solutions

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Instructions. Document your work fully.

For problems 1- 2 prove one using Mathematical Induction.

1. $2 + 4 + 6 + \ldots + 2n = n^2 + n$
   i.e. $\sum_{i=1}^{n} 2i = n^2 + n$

   Proof.

   i. Show that the proposition is true for $n = 1$.

      $\sum_{i=1}^{1} 2i = 2(1) = 2 = (1)^2 + (1)$ True.

   ii. Assume that the proposition is true for $n = k,$ and prove that the proposition is true for $n = k + 1$.

      i.e., Assume that $\sum_{i=1}^{k} 2i = k^2 + k$ for some natural number k, and show that $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

      Observe:

      $$\sum_{i=1}^{k+1} 2i = \underbrace{\sum_{i=1}^{k} 2i + 2(k+1) = \left(k^2 + k\right) + 2(k+1)}_{\text{by Induction Hypothesis}}$$

      $$= k(k+1) + 2(k+1) = (k+2)(k+1)$$

      $$= (k+1)(k+2) = (k+1)[(k+1)+1]$$

      $$= (k+1)^2 + (k+1)$$

      i.e., $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

      Hence, $\sum_{i=1}^{n} 2i = n^2 + n$ for all natural numbers, n. ■

2. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

i.e. $\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$

Proof.

i. Show true for $n = 1$

$\sum_{j=1}^{1} \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{(1)(3)} = \frac{1}{3} = \frac{1}{2(1)+1}$

ii. Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} = \frac{k}{2k+1}$ for some natural number k, and show that $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2(k+1)+1}$

i.e., $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2k+3}$

Observe:

$$\begin{aligned} \sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} &= \sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \text{ (by Induction Hypothesis)} \\ &= \frac{k}{2k+1} \cdot \frac{2k+3}{2k+3} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{(2k+3)} \end{aligned}$$

i.e., $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2k+3}$

Hence, $\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$ for all natural numbers, n. ■

For problems 3- 5 prove one using Mathematical Induction.

3. $2 + 6 + 10 + \ldots + 4n - 2 = 2n^2$

   i.e. $\sum_{i=1}^{n} (4i - 2) = 2n^2$

   Proof.

   i. Show that the proposition is true for $n = 1$.

      $\sum_{i=1}^{1} (4i - 2) = (4(1) - 2) = 2 = 2(1)^2$ True.

   ii. Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

      i.e., Assume that $\sum_{i=1}^{k} (4i - 2) = 2k^2$ for some natural number k, and show that $\sum_{i=1}^{k+1} (4i - 2) = 2(k+1)^2$

      Observe:

      $$\sum_{i=1}^{k+1} (4i - 2) = \underbrace{\sum_{i=1}^{k} (4i - 2) + (4(k+1) - 1) = 2k^2 + (4(k+1) - 2)}_{\text{by Induction Hypothesis}}$$

      $$= 2k^2 + 4k + 2 = 2(k^2 + 2k + 1) = 2(k+1)^2$$

      i.e., $\sum_{i=1}^{k+1} (4i - 2) = 2(k+1)^2$

      Hence, $\sum_{i=1}^{n} (4i - 2) = 2n^2$ for all natural numbers, n. ■

4. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

i.e. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Proof.

i. Show true for $n = 1$

$\sum_{i=1}^{1} i^3 = (1)^3 = 1 = \frac{(1)^2((1)+1)^2}{4}$

ii. Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ for some natural number k, and show that $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

i.e., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$

Observe:

$$\sum_{i=1}^{k+1} i^3 = \underbrace{\sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right] = \frac{(k+1)^2(k+2)^2}{4}$

i.e., $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers, n. ■

5. $\frac{n^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + n^3$ all natural numbers, n.

Proof.

i. Show true for $n = 1$

$\frac{1^4}{4} < 1 = 1^3$

ii. Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\frac{k^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + k^3$ for some natural number k,

and show that $\frac{(k+1)^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + (k+1)^3$

Observe:

$$\underbrace{1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 > \frac{k^4}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^4}{4} + \frac{4(k+1)^3}{4} = \frac{k^4+4k^3+12k^2+12k+4}{4}$$

$> \frac{k^4+4k^3+6k^2+4k+1}{4} = \frac{(k+1)^4}{4}$

i.e., $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 > \frac{(k+1)^4}{4}$

Hence, $\frac{n^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + n^3$ for all natural numbers, n. ■

For problems 6 - 7, prove one using Mathematical Induction:

6. For $0 \le a \le b$; prove that $a^n \le b^n$.

Proof.

i. Show true for $n = 1$.

$$a^1 = \underbrace{a \le b}_{\text{given}} = b^1$$

i.e., $a^1 \le b^1$

ii. Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $a^k \le b^k$ for some natural number k, and show that $a^{k+1} \le b^{k+1}$

Observe:

$$a^{k+1} = a^k \cdot a = \underbrace{b^k \cdot a}_{\text{by Ind. Hyp.}} = \underbrace{b^k \cdot b}_{a \le b} = b^{k+1}$$

i.e., $a^{k+1} \le b^{k+1}$

Hence, $a^n \le b^n$ for all natural numbers, n. ■

7. Given that $\frac{d}{dx}\left[x^0\right] = 0$ and $\frac{d}{dx}\left[x^1\right] = 1$, prove that $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$. You may use the product rule: $\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + g'(x)\,f(x)$.

Proof.

i. Show true for $n = 1$

$$\underbrace{\frac{d}{dx}\left[x^1\right] = 1}_{\text{given}} = 1 \cdot x^{1-1}$$

i.e., $\frac{d}{dx}\left[x^1\right] = 1 \cdot x^{1-1}$.

ii. Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ for some natural number k, and show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^{(k+1)-1}$

re-written: $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Observe:

$$\frac{d}{dx}\left[x^{k+1}\right] = \underbrace{\frac{d}{dx}\left[x^k \cdot x\right] = \left(\frac{d}{dx}\left[x^k\right]\right)x + \left(\frac{d}{dx}\left[x\right]\right)x^k}_{\text{Product Rule}}$$

$$= \underbrace{kx^{k-1}}_{\text{by Ind Hyp}} \cdot x + \underbrace{1}_{\text{given}} \cdot x^k = kx^k + x^k = (k+1)\,x^k$$

i.e., $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Hence, $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ for all natural numbers, n. ■

For problems 8 - 9, prove one using Mathematical Induction:

8. $(1+x)^n \geq 1+nx$ for any natural number n and any real number $x \geq -1$.

Proof.

i. Show true for $n = 1$

$(1+x)^1 = 1+x \geq 1+(1)\,x$

ii. Assume true for $n = k$, and show true for $n = k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number k, and show that $(1+x)^{k+1} \geq 1+(k+1)\,x$

Observe:

$$(1+x)^{k+1} = \underbrace{(1+x)^k (1+x) \geq (1+kx)(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)\,x + \underbrace{kx^2}_{kx^2 \geq 0} \geq 1+(k+1)\,x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)\,x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers n and any real number $x \geq -1$ ■

Our proof hinged on two subtle points:

First, since k is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers x, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

9. Given that $|x_1 + x_2| \leq |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that: $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1|+|x_2|+|x_3|+\ldots+|x_n|$ (the General Triangle Inequality).

Proof.

i. Show that the proposition is true for $n = 1$.

$|x_1| \leq |x_1|$. True.

ii. Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $|x_1 + x_2 + x_3 + \ldots + x_k| \leq |x_1|+|x_2|+|x_3|+\ldots+|x_k|$ and show that

$|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Observe:

$$\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \leq |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from given}}$$

$$\leq \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|}_{\text{by Ind. Hyp.}}.$$

i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all nat. numbers, n. ■