MTH 3311 Test #3

SPRING 2022

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Show CLEARLY how you arrive at your answers.

1. Solve the Differential Equation: $y'' - 3y' - 10y = 14e^{-2x}$

   First, we find the Complementary Solution (i.e., the solution of the related equation: $y'' - 3y' - 10y = 0$)

   $y'' - 3y' - 10y = 0$

   $\Rightarrow \underbrace{\left(D^2 - 3D - 10\right)}_{\phi(D)} y = 0$

   $\Rightarrow \underbrace{m^2 - 3m - 10}_{\phi(m)} = 0$ (This is our auxiliary equation)

   $\Rightarrow (m+2)(m-5) = 0$

   $\Rightarrow m_1 = -2$ and $m_2 = 5$

   $\Rightarrow y_c = C_1 e^{-2x} + C_2 e^{5x}$ (This is our complementary solution)

   Next, we find the particular solution.

   Since the right hand side of the equation is an exponential function, we can use the Method of Undetermined Coefficients.

   Since the right hand side of the equation is $14e^{-2x}$, we suspect that the particular solution is of the form: $y_p = Ce^{-2x}$

   OOPS! This is one of our independent Complementary Solutions

   To get a particular solution that yields $14e^{-2x}$, we multiply our proposed particular solutions by x.

   i.e., $y_p = Cxe^{-2x}$

Next, we'll compute y_p' and y_p'' and plug y_p, y_p', and y_p'' into the equation $y'' - 3y' - 10y = 14e^{-2x}$ to solve for C.

$y_p = Cxe^{-2x}$

$y_p' = \frac{d}{dx}\left[Cxe^{-2x}\right] = \left[Ce^{-2x} - 2xCe^{-2x}\right]$

$y_p'' = \frac{d}{dx}\left[Ce^{-2x} - 2xCe^{-2x}\right] = \left[4xCe^{-2x} - 4Ce^{-2x}\right]$

$y_p'' - 3y_p' - 10y_p = \left(4xCe^{-2x} - 4Ce^{-2x}\right) - 3\left(Ce^{-2x} - 2xCe^{-2x}\right) - 10\left(Cxe^{-2x}\right) = 14e^{-2x}$

Simplifying, we have:

$-7Ce^{-2x} = 14e^{-2x}$

$\Rightarrow C = -2$

Our Particular Solution is $y_p = -2xe^{-2x}$

Our General Solution is: $y = \underbrace{-2xe^{-2x}}_{y_p} + \underbrace{C_1e^{-2x} + C_2e^{5x}}_{y_c}$

$$\boxed{y = -2xe^{-2x} + C_1e^{-2x} + C_2e^{5x}}$$

2. Solve the Differential Equation: $5y'' - 8y' + 5y = 15x^2 - 28x - 27$

First, we find the Complementary Solution (i.e., the solution of the related equation: $5y'' - 8y' + 5y = 0$)

$5y'' - 8y' + 5y = 0$

$\Rightarrow \underbrace{(5D^2 - 8D + 5)}_{\phi(D)} y = 0$

$\Rightarrow \underbrace{5m^2 - 8m + 5}_{\phi(m)} = 0$ (This is our auxiliary equation)

$\Rightarrow m = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(5)(5)}}{2(5)} = \frac{8 \pm \sqrt{-36}}{10} = \frac{8 \pm 6i}{10} = \frac{4}{5} \pm \frac{3}{5}i$

$\Rightarrow m_1 = \frac{4}{5} + \frac{3}{5}i$ and $m_2 = \frac{4}{5} - \frac{3}{5}i$

$\Rightarrow y_c = C_1 e^{\left(\frac{4}{5} + \frac{3}{5}i\right)x} + C_2 e^{\left(\frac{4}{5} - \frac{3}{5}i\right)x}$ (This is our complementary solution)

But - we should rewrite it without using complex coefficients.

$y_c = C_1 e^{\left(\frac{4}{5} + \frac{3}{5}i\right)x} + C_2 e^{\left(\frac{4}{5} - \frac{3}{5}i\right)x} = C_1 e^{\frac{4}{5}x + \frac{3}{5}ix} + C_2 e^{\frac{4}{5}x - \frac{3}{5}ix} = C_1 e^{\frac{4}{5}x} e^{i\frac{3}{5}x} + C_2 e^{\frac{4}{5}x} e^{-i\frac{3}{5}x}$

$= e^{\frac{4}{5}x} \left(C_1 e^{i\frac{3}{5}x} + C_2 e^{-i\frac{3}{5}x} \right) = e^{\frac{4}{5}x} \left(A \cos\left(\frac{3}{5}x\right) + B \sin\left(\frac{3}{5}x\right) \right)$

i.e. $y_c = e^{\frac{4}{5}x} \left(A \cos\left(\frac{3}{5}x\right) + B \sin\left(\frac{3}{5}x\right) \right)$ (This is our complementary solution)

Remark: We used the identity: $C_1 e^{im_1 x} + C_2 e^{-im_2 x} = (A \cos(m_1 x) + B \sin(m_2 x))$

Next, we find the particular solution.

Since the right hand side of the equation is a polynomial, we can use the Method of Undetermined Coefficients.

Since the right hand side of the equation is $15x^2 - 28x - 27$, we suspect that the particular solution is of the form: $y_p = C_1 x^2 + C_2 x + C_3$

Next, we'll compute y_p' and y_p'' and plug y_p, y_p', and y_p'' into the equation $5y''-8y'+5y = 15x^2-28x-27$ to solve for C.

$y_p = C_1x^2 + C_2x + C_3$

$y_p' = 2C_1x + C_2$

$y_p'' = 2C_1$

$5y'' - 8y' + 5y = 5\left(2C_1\right) - 8\left(2C_1x + C_2\right) + 5\left(C_1x^2 + C_2x + C_3\right) = 15x^2 - 28x - 27$

$\Rightarrow 5C_1x^2 + \left(-16C_1 + 5C_2\right)x + \left(10C_1 - 8C_2 + 5C_3\right) = 15x^2 - 28x - 27$

Comparing coefficients of x, we have:

$5C_1 = 15 \Rightarrow C_1 = 3$

$$\boxed{C_1 = 3}$$

$-16C_1 + 5C_2 = -28 \Rightarrow -16\left(3\right) + 5C_2 = -28 \Rightarrow 5C_2 = 20 \Rightarrow C_2 = 4$

$$\boxed{C_2 = 4}$$

$\left(10C_1 - 8C_2 + 5C_3\right) = -27 \Rightarrow \left(10\left(3\right) - 8\left(4\right) + 5C_3\right) = -27 \Rightarrow 5C_3 = -25 \Rightarrow C_3 = -5$

$$\boxed{C_3 = -5}$$

Our Particular Solution is $y_p = C_1x^2 + C_2x + C_3 = 3x^2 + 4x - 5$

i.e., $y_p = 3x^2 + 4x - 5$

Our General Solution is: $y = \underbrace{3x^2 + 4x - 5}_{y_p} + \underbrace{\left(A\cos\left(\dfrac{3}{5}x\right) + B\sin\left(\dfrac{3}{5}x\right)\right)}_{y_c}$

$$\boxed{y = 3x^2 + 4x - 5 + A\cos\left(\tfrac{3}{5}x\right) + B\sin\left(\tfrac{3}{5}x\right)}$$

3. Solve the Differential Equation: $y'' - 2y' + y = \frac{e^x}{x^2}$

First, we find the Complementary Solution (i.e., the solution of the related equation: $y'' - 2y' + y = 0$)

$y'' - 2y' + y = 0$

$\Rightarrow \underbrace{\left(D^2 - 2D + 1\right)}_{\phi(D)} y = 0$

$\Rightarrow \underbrace{m^2 - 2m + 1}_{\phi(m)} = 0$ (This is our auxiliary equation)

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m_1 = 1 = m_2 = 1$

This yields solutions of $C_1 e^x$ and $C_2 e^x$

OOPS! These solutions are one and the same.

To get two independent solutions, we multiply one of them by x.

Our two independent solutions are $C_1 e^x$ and $C_2 x e^x$

$\Rightarrow y_c = C_1 e^x + C_2 x e^x$ (This is our complementary solution)

Next, we find the particular solution.

Since the right hand side of the equation is NOT a linear combination of polynomials, exponentials, or sines and cosines, we CAN'T use the Method of Undetermined Coefficients.

We have to use Variation of Parameters.

This means that we must vary the parameters C_1 and C_2 byt converting them to functions of x. Namely, $A(x)$ and $B(x)$.

Our proposed general solution is:

$y = A(x) e^x + B(x) x e^x$

We impose two restrictions on the pair of functions $A(x)$ and $B(x)$.

The first restriction is that the pair $A(x)$ and $B(x)$ are such that $y = A(x) e^x + B(x) x e^x$ is the general solution of our equation.

There are infinitely many pairs of functions that satisfy this condition.

The second restriction, which we hold in abeyance for the time being, will uniquely define the pair $A(x)$ and $B(x)$.

Given $y = A(x)e^x + B(x)xe^x$, we compute y' and y''.

$y = A(x)e^x + B(x)xe^x$

$y' = A'(x)e^x + A(x)e^x + B'(x)xe^x + B(x)e^x + B(x)xe^x$

We now impose our second restriction – namely that $A(x)$ and $B(x)$ are such that

$A'(x)e^x + B'(x)xe^x = 0.$

This restriction has the advantage that $A'(x)$ and $B'(x)$ are eliminated from the definition of y'

Thus, we have:

$y = A(x)e^x + B(x)xe^x$

$y' = A(x)e^x + B(x)xe^x + B(x)e^x$

$y'' = A'(x)e^x + A(x)e^x + B'(x)xe^x + B(x)e^x + B(x)xe^x + B'(x)e^x + B(x)e^x$

(Combining "like terms" and applying our second restriction to y'', we have:

$y'' = A(x)e^x + 2B(x)e^x + B(x)xe^x + B'(x)e^x$

Next, we plug these definitions of y, y', y'' into the original equation: $y'' - 2y' + y = \frac{e^x}{x^2}$

$$\begin{array}{lllllllll}
y'' & = & A(x)e^x & + & 2B(x)e^x & + & B(x)xe^x & + & B'(x)e^x \\
-2y' & = & -2A(x)e^x & - & 2B(x)e^x & - & 2B(x)xe^x & & \\
+y & = & A(x)e^x & & & + & B(x)xe^x & & \\
\hline
y'' - 2y' + y & = & & & & & & & B'(x)e^x \quad = \quad \frac{e^x}{x^2}
\end{array}$$

$\Rightarrow B'(x)e^x = \frac{e^x}{x^2}$

$\Rightarrow B'(x) = \frac{1}{x^2}$

$\Rightarrow B(x) = -\frac{1}{x} + C_2$

To find $A(x)$, we plug $B'(x) = \frac{1}{x^2}$ into our second restriction: $A'(x)e^x + B'(x)xe^x = 0$.

$$\Rightarrow A'(x)e^x + \underbrace{\frac{1}{x^2}}_{B'(x)} xe^x = 0$$

$$\Rightarrow A'(x)e^x + \tfrac{1}{x}e^x = 0$$

$$\Rightarrow A'(x) + \tfrac{1}{x} = 0$$

$$\Rightarrow A'(x) = -\tfrac{1}{x}$$

$$\Rightarrow A(x) = -\ln(x) + C_1$$

Thus, our general solution $y = A(x)e^x + B(x)xe^x$ becomes:

$$y = (-\ln(x) + C_1)e^x + \left(-\tfrac{1}{x} + C_2\right)xe^x$$

$$y = -\ln(x)e^x + C_1e^x + e^x + C_2xe^x$$

$$y = -\ln(x)e^x + (C_1 + 1)e^x + C_2xe^x$$

$$\boxed{y = -\ln(x)e^x + C_3e^x + C_2xe^x}$$