MTH 1125 Test #1 - Solutions

Summer 2015

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 2}\frac{x^2+5}{x^2-2} =$

   i) Try Plugging in:

   $\lim_{x\to 2}\frac{x^2+5}{x^2-2} = \frac{(2)^2+5}{(2)^2-2} = \frac{4+5}{4-2} = \frac{9}{2}$

   i.e., $\lim_{x\to 2}\frac{x^2+5}{x^2-2} = \frac{9}{2}$

2. Compute: $\lim_{x\to -1}\frac{x^2+4x+3}{x^2-5x-6} =$

   i) Try Plugging in:

   $\lim_{x\to -1}\frac{x^2+4x+3}{x^2-5x-6} = \frac{(-1)^2+4(-1)+3}{(-1)^2-5(-1)-6} = \frac{1-4+3}{1+5-6} = \frac{0}{0}$ No Good - Zero Divide!

   ii) Try Factoring and Cancelling:

   $\lim_{x\to -1}\frac{x^2+4x+3}{x^2-5x-6} = \lim_{x\to -1}\frac{(x+1)(x+3)}{(x+1)(x-6)} = \lim_{x\to -1}\frac{(x+3)}{(x-6)} = \frac{(-1)+3}{(-1)-6} = \frac{2}{-7} = -\frac{2}{7}$

   i.e., $\lim_{x\to -1}\frac{x^2+4x+3}{x^2-5x-6} = -\frac{2}{7}$

3. Compute: $\lim_{x\to 1}\frac{x-5}{x^2-5x+4} =$

   i) Try Plugging in:

   $\lim_{x\to 1}\frac{x-5}{x^2-5x+4} = \frac{(1)-5}{(1)^2-5(1)+4} = \frac{-4}{0}$ No Good - Zero Divide!

   ii) Try Factoring and Cancelling:

   No Good!. "Factoring and Cancelling" only works when Step #1 yields $\frac{0}{0}$.

iii) Analyze the one-sided limits:

$$\lim_{x\to 1^-} \frac{x-5}{x^2-5x+4} = \lim_{x\to 1^-} \frac{x-5}{(x-1)(x-4)} = \frac{-4}{(-\varepsilon)(-3)} = \frac{\left(\frac{-4}{-3}\right)}{(-\varepsilon)} = \frac{\left(\frac{4}{3}\right)}{(-\varepsilon)} = -\infty$$

$$\begin{array}{|ll|} & x \to 1^- \\ \Rightarrow & x < 1 \\ \Rightarrow & x - 1 < 0 \end{array}$$

$$\lim_{x\to 1^+} \frac{x-5}{x^2-5x+4} = \lim_{x\to 1^+} \frac{x-5}{(x-1)(x-4)} = \frac{-4}{(+\varepsilon)(-3)} = \frac{\left(\frac{-4}{-3}\right)}{(+\varepsilon)} = \frac{\left(\frac{4}{3}\right)}{(+\varepsilon)} = +\infty$$

$$\begin{array}{|ll|} & x \to 1^+ \\ \Rightarrow & x > 1 \\ \Rightarrow & x - 1 > 0 \end{array}$$

Since the one-sided limits are not equal, $\lim_{x\to 1} \frac{x-5}{x^2-5x+4}$ **Does Not Exist!**

4. Determine whether or not $f(x)$ is continuous at the point $x = 3$. (Justify your answer.)

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x < 3 \\ 9 & \text{for } x = 3 \\ 3x - 3 & \text{for } x > 3 \end{cases}$$

If $f(x)$ is continuous at the point $x = 3$, then $\lim_{x\to 3} f(x) = f(3)$.

To see if this is true, we'll compute $\lim_{x\to 3} f(x)$.

Since the definition of $f(x)$ for $x < 3$ is different than the definition of $f(x)$ for $x > 3$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3^-} (x+3) = 6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3x-3) = 3(3) - 3 = 6$

Since the one-sided limits ARE EQUAL, $\lim_{x\to 3} f(x)$ EXISTS and is equal to the common value of the one-sided limits.

i.e., $\lim_{x\to 3} f(x) = 6$

However, $f(3) = 9$

Therefore, since $\lim_{x\to 3} f(x) \neq f(3)$,

$f(x)$ is NOT continuous at $x = 3$

5. Find the asymptotes and graph: $f(x) = \frac{x^2+x-6}{x^2-x-6}$

$\boxed{\text{Verticals}}$

1. Find x-values that cause division by zero.

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x+2)(x-3) = 0$

$\Rightarrow x = -2$ and $x = 3$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$$\lim_{x\to -2^-} \frac{x^2+x-6}{x^2-x-6} = \lim_{x\to -2^-} \frac{x^2+x-6}{(x+2)(x-3)} = \frac{-4}{(-\varepsilon)(-5)} = \frac{\left(\frac{-4}{-5}\right)}{(-\varepsilon)} = \frac{\left(\frac{4}{5}\right)}{-\varepsilon} = -\infty$$

$$\boxed{\begin{array}{ll} & x \to -2^- \\ \Rightarrow & x < -2 \\ \Rightarrow & x+2 < 0 \end{array}}$$

$$\lim_{x\to -2^+} \frac{x^2+x-6}{x^2-x-6} = \lim_{x\to -2^+} \frac{x^2+x-6}{(x+2)(x-3)} = \frac{-4}{(\varepsilon)(-5)} = \frac{\left(\frac{-4}{-5}\right)}{(\varepsilon)} = \frac{\left(\frac{4}{5}\right)}{\varepsilon} = +\infty$$

$$\boxed{\begin{array}{ll} & x \to -2^+ \\ \Rightarrow & x > -2 \\ \Rightarrow & x+2 > 0 \end{array}}$$

Since the one-sided limits are infinite, $x = -2$ *IS a vertical asymptote.*

$$\lim_{x\to 3^-} \frac{x^2+x-6}{x^2-x-6} = \lim_{x\to 3^-} \frac{x^2+x-6}{(x+2)(x-3)} = \frac{6}{(5)(-\varepsilon)} = \frac{\left(\frac{6}{5}\right)}{(-\varepsilon)} = -\infty$$

$$\boxed{\begin{array}{ll} & x \to 3^- \\ \Rightarrow & x < 3 \\ \Rightarrow & x-3 < 0 \end{array}}$$

$$\lim_{x\to 3^+} \frac{x^2+x-6}{x^2-x-6} = \lim_{x\to 3^+} \frac{x^2+x-6}{(x+2)(x-3)} = \frac{6}{(5)(+\varepsilon)} = \frac{\left(\frac{6}{5}\right)}{(+\varepsilon)} = +\infty$$

$$\boxed{\begin{array}{ll} & x \to 3^+ \\ \Rightarrow & x > 3 \\ \Rightarrow & x-3 > 0 \end{array}}$$

Since the one-sided limits are infinite, $x = 3$ *IS a vertical asymptote.*

Horizontals

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to-\infty} \frac{x^2+x-6}{x^2-x-6} = \lim_{x\to-\infty} \frac{x^2}{x^2} = \lim_{x\to-\infty} 1 = 1$

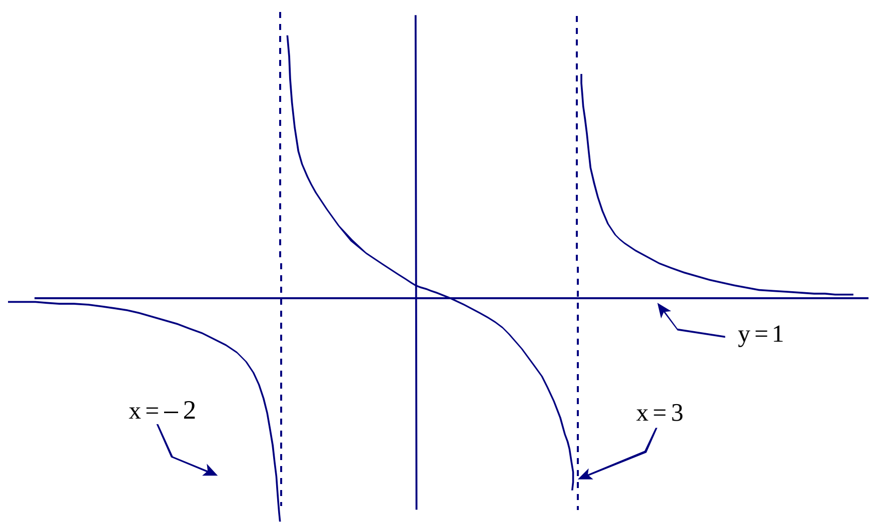
$\lim_{x\to+\infty} \frac{x^2+x-6}{x^2-x-6} = \lim_{x\to+\infty} \frac{x^2}{x^2} = \lim_{x\to+\infty} 1 = 1$

Since the limits are finite and constant, $y = 1$ is a horizontal asymptote.

Summary:

$\lim_{x\to-2^-} \frac{x^2+x-6}{x^2-x-6} = -\infty$	
$\lim_{x\to-2^+} \frac{x^2+x-6}{x^2-x-6} = +\infty$	$\lim_{x\to-\infty} \frac{x^2+x-6}{x^2-x-6} = 1$
$\lim_{x\to3^-} \frac{x^2+x-6}{x^2-x-6} = -\infty$	$\lim_{x\to+\infty} \frac{x^2+x-6}{x^2-x-6} = 1$
$\lim_{x\to3^+} \frac{x^2+x-6}{x^2-x-6} = +\infty$	

Graph $f(x) = \frac{x^2+x-6}{x^2-x-6}$



6. Compute: $\lim_{x\to 4} \frac{\sqrt{12+x}-4}{x-4} =$

i) Try Plugging in:

$$\lim_{x\to 4} \frac{\sqrt{12+x}-4}{x-4} = \frac{\sqrt{12+(4)}-5}{(4)-4} = \frac{0}{0}$$

No Good -
Zero Divide!

ii) Try Factoring and Cancelling:

$$\lim_{x\to 4} \frac{\sqrt{12+x}-4}{x-4} = \lim_{x\to 4} \frac{\sqrt{12+x}-4}{x-4} \cdot \frac{\sqrt{12+x}+4}{\sqrt{12+x}+4} = \lim_{x\to 4} \frac{\left(\sqrt{12+x}\right)^2-(4)^2}{(x-4)\left[\sqrt{12+x}+4\right]}$$

$$= \lim_{x\to 4} \frac{12+x-16}{(x-4)\left[\sqrt{12+x}+4\right]} = \lim_{x\to 4} \frac{(x-4)}{(x-4)\left[\sqrt{12+x}+4\right]} = \lim_{x\to 4} \frac{1}{\sqrt{12+x}+4}$$

$$\frac{1}{\sqrt{12+(4)}+4} = \frac{1}{4+4} = \frac{1}{8}$$

$$\boxed{\text{i.e., } \lim_{x\to 4} \frac{\sqrt{12+x}-4}{x-4} = \frac{1}{8}}$$

7.

$x =$	$f(x) =$
-10.1	2.5
-100.8	2.9
$-1,000.3$	2.99
$-10,000.3$	2.999
$-100,000.9$	2.9999

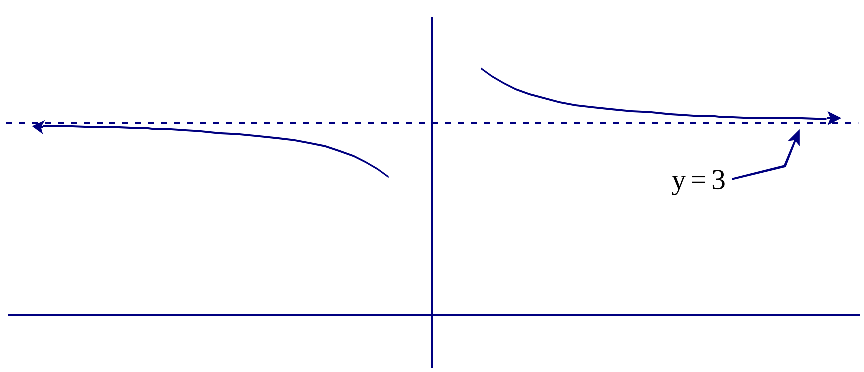
$x =$	$f(x) =$
10.1	3.5
100.8	3.1
$1,000.3$	3.01
$10,000.3$	3.001
$100,000.9$	3.0001

Based on the information in the table above, do the following:

(a) $\lim_{x\to -\infty} f(x) = 3$

(b) $\lim_{x\to +\infty} f(x) = 3$

(c) Graph $f(x)$



8. Compute: $\lim_{x\to -\infty} \frac{3x^5+4x-8x}{9x^4-8x^2-5} =$

$\lim_{x\to -\infty} \frac{3x^5+4x-8x}{9x^4-8x^2-5} = \lim_{x\to -\infty} \frac{3x^5}{9x^4} = \lim_{x\to -\infty} \frac{x}{3} = -\infty$

i.e., $\lim_{x\to -\infty} \frac{3x^5+4x-8x}{9x^4-8x^2-5} = -\infty$

9. $f(x) = 3x^6 - 3x^5 + 4x^3 - 6x^2 + 9x - 5$;   Compute: $f'(x)$.

$f'(x) = 3(6x^5) - 3(5x^4) + 4(3x^2) - 6(2x^1) + 9(1) - 0 = 18x^5 - 15x^4 + 12x^2 - 12x + 9$

i.e., $f'(x) = 18x^5 - 15x^4 + 12x^2 - 12x + 9$

10. $\frac{d}{dx}[8\sin(x) + 4\cos(x)] =$

$\frac{d}{dx}[8\sin(x) + 4\cos(x)] = 8(\cos(x)) + 4(-\sin(x)) = 8\cos(x) - 4\sin(x)$

i.e., $\frac{d}{dx}[8\sin(x) + 4\cos(x)] = 8\cos(x) - 4\sin(x)$

11. $f(x) = 6x^2 - 2x$; compute $f'(x)$ **using the definition of derivative.** (i.e., compute $f'(x)$ using the "limiting process.")

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[6(x+\Delta x)^2-2(x+\Delta x)\right]-\left(6x^2-2x\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[6\left(x^2+2x\Delta x+\Delta x^2\right)-2(x+\Delta x)\right]-\left(6x^2-2x\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[\left(6x^2+12x\Delta x+6\Delta x^2\right)-(2x+2\Delta x)\right]-\left(6x^2-2x\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{12x\Delta x+6\Delta x^2-2\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\Delta x(12x+6\Delta x-2)}{\Delta x} = \lim_{\Delta x\to 0}(12x + 6\Delta x - 2) = 12x + 6(0) - 2 = 12x - 2$

i.e., $f'(x) = 12x - 2$

12. Compute: $\frac{d}{dx}\left[x^3 \sin(x)\right] =$

$$\frac{d}{dx}\left[\underbrace{x^3}_{1^{\text{st}}} \cdot \underbrace{\sin(x)}_{2^{\text{nd}}}\right] = \underbrace{3x^2}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{\sin(x)}_{2^{\text{nd}}} + \underbrace{\cos(x)}_{2^{\text{nd}}\text{prime}} \cdot \underbrace{x^3}_{1^{\text{st}}} = 3x^2 \sin(x) + x^3 \cos(x)$$

i.e., $\frac{d}{dx}\left[x^3 \sin(x)\right] = 3x^2 \sin(x) + x^3 \cos(x)$

13. Compute: $\frac{d}{dx}\left[\frac{\cos(x)}{4x^2+2x}\right] =$

$$\frac{d}{dx}\left[\frac{\overbrace{\cos(x)}^{\text{top}}}{\underbrace{4x^2+2x}_{\text{bottom}}}\right] = \frac{\overbrace{(-\sin(x))}^{\text{top prime}} \cdot \overbrace{\left(4x^2+2x\right)}^{\text{bottom}} - \overbrace{(8x+2)}^{\text{bottom prime}} \cdot \overbrace{\cos(x)}^{\text{top}}}{\underbrace{\left(4x^2+2x\right)^2}_{\text{bottom squared}}}$$

i.e., $\frac{d}{dx}\left[\frac{\cos(x)}{4x^2+2x}\right] = -\frac{\sin(x)\left(4x^2+2x\right) + (8x+2)\cos(x)}{\left(4x^2+2x\right)^2}$