

# Formulas You Should Memorize (and I do mean Memorize!)

SPRING 2022

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## Formulas From Calculus

1.  $\frac{d}{dx} [\sin(x)] = \cos(x)$
2.  $\frac{d}{dx} [\cos(x)] = -\sin(x)$
3.  $\frac{d}{dx} [\tan(x)] = \sec^2(x)$
4.  $\frac{d}{dx} [\cot(x)] = -\csc^2(x)$
5.  $\frac{d}{dx} [\sec(x)] = \sec(x) \tan(x)$
6.  $\frac{d}{dx} [\csc(x)] = -\csc(x) \cot(x)$

**Remark 1** Note that if you know the derivatives of  $\sin(x)$ ,  $\tan(x)$ , and  $\sec(x)$ , then the derivatives of the corresponding “co-functions”  $\cos(x)$ ,  $\cot(x)$ , and  $\csc(x)$  are found by:

(a) Changing the sign, and

(b) Replacing each factor of the derivative with its co-function.

**Example 1**  $\frac{d}{dx} [\sec(x)] = \sec(x) \tan(x) \implies \frac{d}{dx} [\csc(x)] = -\csc(x) \cot(x)$

7.  $\int \sin(x) dx = -\cos(x) + C$
8.  $\int \cos(x) dx = \sin(x) + C$
9.  $\int \tan(x) dx = \ln |\sec(x)| + C = -\ln |\cos(x)| + C$
10.  $\int \cot(x) dx = \ln |\sin(x)| + C$
11.  $\int \sec(x) dx = \ln |\sec(x) + \tan(x)| + C$
12.  $\int \csc(x) dx = \ln |\csc(x) - \cot(x)| + C$
13.  $\int \sec^2(x) dx = \tan(x) + C$
14.  $\int \csc^2(x) dx = -\cot(x) + C$
15.  $\int \sec(x) \tan(x) dx = \sec(x) + C$
16.  $\int \csc(x) \cot(x) dx = -\csc(x) + C$
17.  $\frac{d}{dx} [e^u] = e^u \cdot \frac{du}{dx}$
18.  $\frac{d}{dx} [\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$

19.  $\int e^u du = e^u + C$
20.  $\int \frac{1}{u} du = \ln |u| + C$
21.  $\int u^{-1} du = \ln |u| + C$
22.  $\int \ln(u) du = u \ln(u) - u + C$
23.  $\frac{d}{dx} [\sin^{-1}(u)] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
24.  $\frac{d}{dx} [\cos^{-1}(u)] = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
25.  $\frac{d}{dx} [\tan^{-1}(u)] = \frac{1}{1+u^2} \cdot \frac{du}{dx}$
26.  $\frac{d}{dx} [\cot^{-1}(u)] = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$
27.  $\frac{d}{dx} [\sec^{-1}(u)] = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$
28.  $\frac{d}{dx} [\csc^{-1}(u)] = -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

**Remark 2** Note that if you know the derivatives of the inverse trig functions  $\sin^{-1}(x)$ ,  $\tan^{-1}(x)$ , and  $\sec^{-1}(x)$ , the derivatives of the corresponding inverse “co-functions” can be found by changing the sign.

29.  $\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C$
30.  $\int \frac{1}{a^2+u^2} du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$
31.  $\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a} \sec^{-1}\left(\frac{|u|}{a}\right) + C$

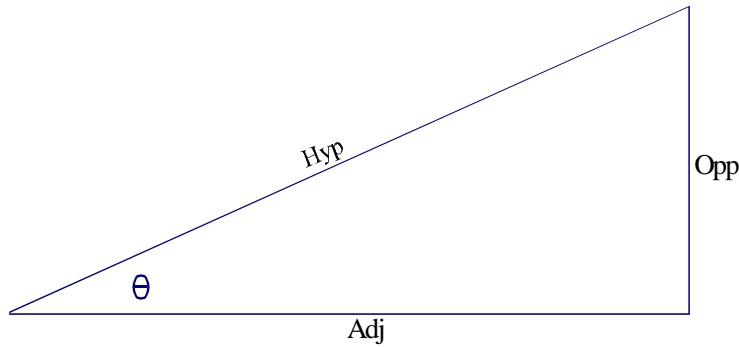
## Essential Trigonometric Identities

32.  $\sin^2(x) + \cos^2(x) = 1$
33.  $\tan^2(x) + 1 = \sec^2(x)$
34.  $\cot^2(x) + 1 = \csc^2(x)$

**Remark 3** Identities 32-34 are the so-called “Pythagorean Identities”.

35.  $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{\cot(x)}$
36.  $\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$
37.  $\sec(x) = \frac{1}{\cos(x)}$
38.  $\csc(x) = \frac{1}{\sin(x)}$
39.  $\sin(2x) = 2 \sin(x) \cos(x)$
40.  $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2 \sin^2(x) = 2 \cos^2(x) - 1$

## Right Triangle Trig.



$$41. \sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$

$$42. \cos(\theta) = \frac{\text{adj}}{\text{hyp}}$$

$$43. \tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$44. \cot(\theta) = \frac{\text{adj}}{\text{opp}}$$

$$45. \sec(\theta) = \frac{\text{hyp}}{\text{adj}}$$

$$46. \csc(\theta) = \frac{\text{hyp}}{\text{opp}}$$

## Logs and Exponential Functions

$$47. \ln(xy) = \ln(x) + \ln(y)$$

$$(a) \log_a(xy) = \log_a(x) + \log_a(y)$$

$$48. \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$(a) \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$49. \ln(x^n) = n \cdot \ln(x)$$

$$(a) \log_a(x^n) = n \cdot \log_a(x)$$

$$50. \ln(e^x) = x$$

$$(a) \log_a(a^x) = x$$

$$51. e^{\ln(x)} = x$$

$$(a) a^{\log_a(x)} = x$$

52.  $\ln(e) = 1$

(a)  $\log_a(a) = 1$

53.  $\ln(1) = 0$

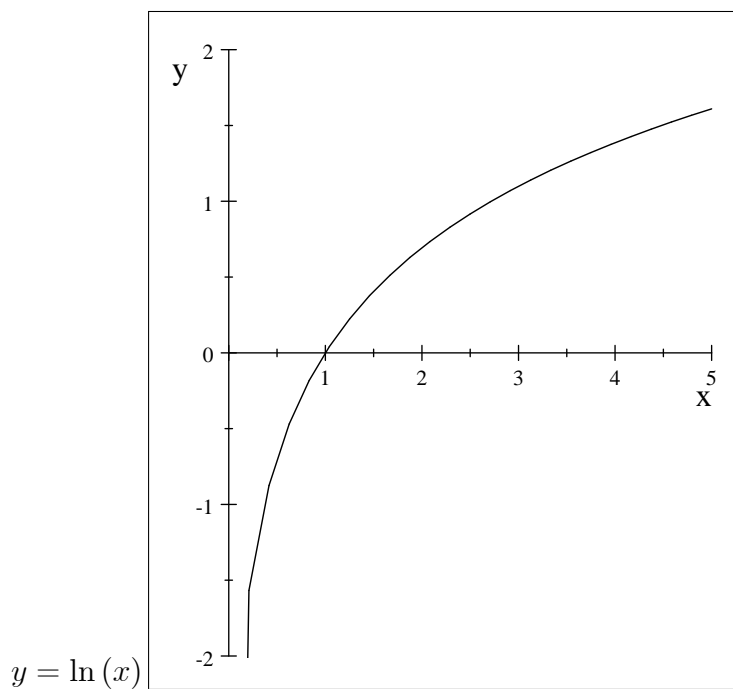
(a)  $\log_a(1) = 0$

54.  $\lim_{x \rightarrow \infty} \ln(x) = \infty$

(a)  $\lim_{x \rightarrow \infty} \log_a(x) = \infty$

55.  $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$

(a)  $\lim_{x \rightarrow 0^+} \log_a(x) = -\infty$



56.  $e^{x+y} = e^x e^y$

(a)  $a^{x+y} = a^x a^y$

57.  $\frac{e^x}{e^y} = e^{x-y}$

(a)  $\frac{a^x}{a^y} = a^{x-y}$

58.  $(e^x)^n = e^{xn}$

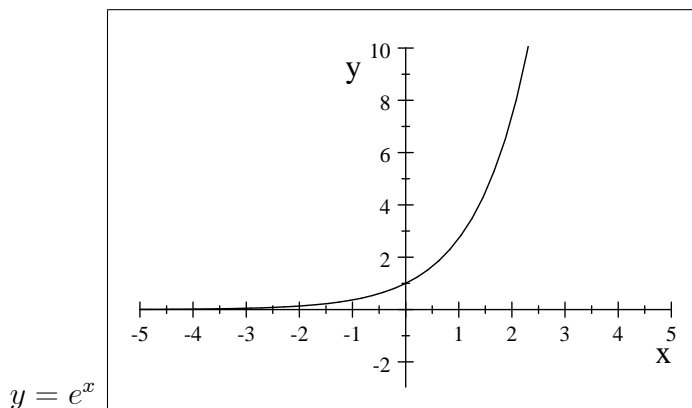
(a)  $(a^x)^n = a^{xn}$

$$59. \lim_{x \rightarrow \infty} e^x = \infty$$

$$(a) \lim_{x \rightarrow \infty} a^x = \infty$$

$$60. \lim_{x \rightarrow -\infty} e^x = 0$$

$$(a) \lim_{x \rightarrow -\infty} a^x = 0$$



$$61. \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$$

$$62. \lim_{x \rightarrow 0} (1 + kx)^{\frac{1}{x}} = e^k$$

$$(a) \lim_{x \rightarrow 0} (1 - kx)^{\frac{1}{x}} = e^{-k}$$

$$63. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$64. \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x = e^k$$

$$(a) \lim_{x \rightarrow \infty} \left(1 - \frac{k}{x}\right)^x = e^{-k}$$

**Remark 4** Note that 62.a and 64.a can be obtained easily from 62 and 64 respectively, by “plugging in”  $-k$  in place of  $k$ .

$$65. (xy)^n = x^n y^n$$

$$66. \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

## Calculus of General Logs and Exponentials

$$67. \frac{d}{dx} [a^x] = \ln(a) \cdot a^x$$

$$68. \frac{d}{dx} [a^u] = \ln(a) \cdot a^u \cdot \frac{du}{dx}$$

$$69. \int a^u du = \frac{1}{\ln(a)} a^u + C$$

$$70. \frac{d}{dx} [\log_a(x)] = \frac{1}{\ln(a)} \cdot \frac{1}{x}$$

$$71. \frac{d}{dx} [\log_a(u)] = \frac{1}{\ln(a)} \cdot \frac{1}{u} \cdot \frac{du}{dx}$$

## By-Passing U-Substitution, Given Simple Composite Functions

Suppose that  $F(x)$  is an anti-derivative of  $f(x)$ . (i.e., suppose that  $\int f(x) dx = F(x) + C$ )

Then if  $k$  and  $c$  are constants, the following general principles hold:

$$\int f(kx) dx = \frac{1}{k}F(kx) + C \quad \text{and} \quad \int f(kx + c) dx = \frac{1}{k}F(kx + c) + C$$

**Some Cases in Point:**

$$\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + C$$

$$\int \sin(kx + c) dx = -\frac{1}{k} \cos(kx + c) + C$$

$$\int \cos(kx) dx = \frac{1}{k} \sin(kx) + C$$

$$\int \cos(kx + c) dx = \frac{1}{k} \sin(kx + c) + C$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C$$

$$\int e^{kx+c} dx = \frac{1}{k} e^{kx+c} + C$$

$$\int \frac{1}{kx} dx = \frac{1}{k} \ln(kx) + C$$

$$\int \frac{1}{kx+c} dx = \frac{1}{k} \ln(kx + c) + C$$

## Series and Sums

### Finite Series

72.  $\sum_{i=0}^n r^i \cdot a = \frac{1-r^{n+1}}{1-r}$  **Finite Geometric Series** with ratio  $r$ .

73.  $\sum_{i=0}^n (ki + a) = \frac{n+1}{2}(a_1 + a_2)$  **Arithmetic Series** with common difference  $k$ .

74.  $\sum_{i=1}^n a_i = \sum_{i=1}^k a_i + \sum_{i=k+1}^n a_i$

### Infinite Series

75. If  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$ . Conversely, if  $a_n$  does not go to 0, then the series must diverge.

76.  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$  is the **Harmonic Series**. It diverges.

77. The series:

(a)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots$  and

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots + \frac{(-1)^n}{n} + \dots$

are **Alternating Harmonic Series**. These series converge.

78.  $\sum_{n=0}^{\infty} r^n a = a + ra + r^2a + r^3a + \dots + r^na + \dots$  is the **Geometric Series** with ratio  $r$ .

- (a) If  $|r| < 1$ , the series converges, and the sum is given by  $\frac{\text{first term}}{1-r}$ .  
 (b) If  $|r| \geq 1$ , the series diverges.

79. The series:

(a)  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n+1} a_n + \dots$  and

(b)  $\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \dots + (-1)^{n+1} a_n + \dots$  where:

1.  $a_n \geq 0$
2.  $a_n \geq a_{n+1}$
3.  $\lim_{n \rightarrow \infty} a_n = 0$

are **Alternating Series**. They converge. Also:

$$\left| \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^k (-1)^{n+1} a_n \right| \leq |a_{k+1}|$$

(A similar statement can be made for  $\sum_{n=1}^{\infty} (-1)^n a_n$ )

80. **Direct Comparison Test** Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be such that  $a_n$  and  $b_n$  are non-negative for all but finitely many terms.

- (a) If  $a_n \leq b_n$  for all but finitely many terms and  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges also.  
 (b) If  $a_n \leq b_n$  for all but finitely many terms and  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  diverges also.  
 (c) If  $a_n \leq b_n$  and  $\sum_{n=1}^{\infty} a_n$  converges, this tells us nothing about  $\sum_{n=1}^{\infty} b_n$ .  
 (d) If  $a_n \leq b_n$  and  $\sum_{n=1}^{\infty} b_n$  diverges, this tells us nothing about  $\sum_{n=1}^{\infty} a_n$ .

81. **Limit Comparison Test** Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be such that  $a_n$  and  $b_n$  are non-negative for all but finitely many terms.

- (a) If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ , where  $c$  is **non-zero and finite** (i.e.  $0 < c < \infty$ ), then both  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  converge, or both diverge.  
 (b) If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  or if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ , we can conclude nothing. In this case, we have made a poor choice for comparison. Choose another series for comparison.

82. **p-series** The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  where  $p$  is a positive constant, is the **p-series**. This series converges if  $p > 1$  and diverges if  $p \leq 1$ .

83. **Ratio Test** Consider  $\sum_{n=1}^{\infty} a_n$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.

(c) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , the Ratio Test is inconclusive.

84. **The Integral Test** Let  $\sum_{n=1}^{\infty} a_n$  be such that  $a_n$  is non-negative for all but finitely many terms. If:

(a)  $f(x)$  is a continuous function on the interval  $[1, \infty)$ , with the property that  $f(n) = a_n$  for  $n = 1, 2, 3, \dots$  and

(b)  $f(x)$  is decreasing on  $[1, \infty)$ ,

then either both  $\sum_{n=1}^{\infty} a_n$  and  $\int_a^b f(x) dx$  converge, or they both diverge.

85. **The  $n^{\text{th}}$  Root Test** Consider  $\sum_{n=1}^{\infty} a_n$ .

(a) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ , then series  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ , then series  $\sum_{n=1}^{\infty} a_n$  diverges.

(c) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , then the  $n^{\text{th}}$  Root Test is inconclusive.

86. The series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$  is the **Taylor Series** of  $f(x)$  with center  $x_0$ . Here,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

## Some Well-Known Taylor Series

87.  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$  for all  $x$ .

88.  $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots$  for  $|x| < 1$

89.  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$  for all  $x$ .

90.  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$  for all  $x$ .

91.  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots + x^n + \dots$  for  $|x| < 1$ .