MTH 2215 Test 2 - Solutions

Spring 2021

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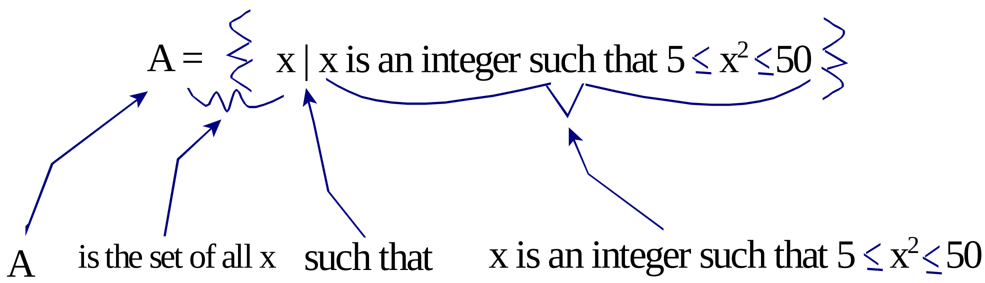
Name ______________________________

Show CLEARLY how you arrive at your answers.

1. List the members of the set: $\{x|\ x$ is an integer such that $5 \leq x^2 \leq 50\}$ in roster form:

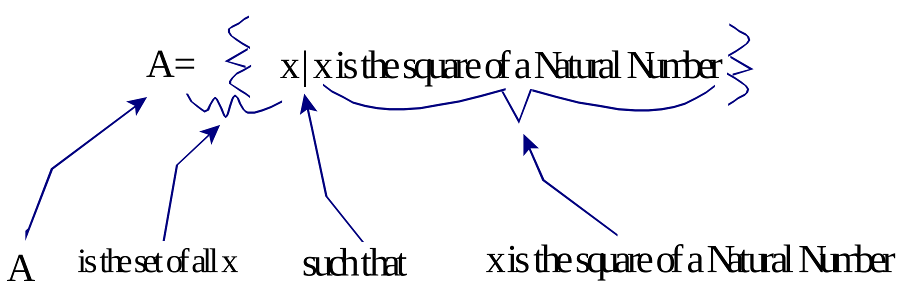


$$A = \{-7, -6, -5, -4, -3, 3, 4, 5, 6, 7\}$$

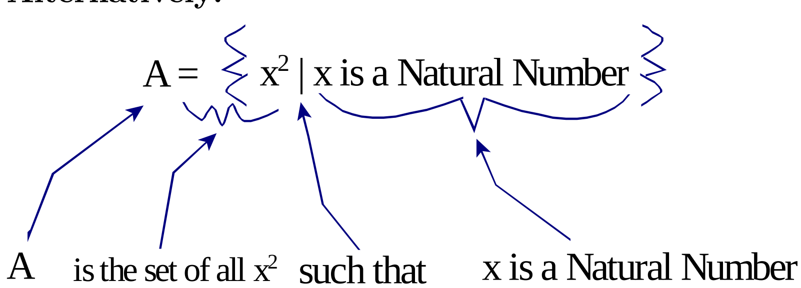
2. Express the set $A = \{0, 1, 4, 9, 16, 25, \ldots\}$ using "set builder notation."

Observe: $A = \{0, 1, 4, 9, 16, 25, \ldots\} = \{0^2, 1^2, 2^2, 3^2, 4^2, 5^2, \ldots\}$

This is the set of the squares of all natural numbers.



Alternatively:



3. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. Compute $A \times B$

   $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

   Therefore, $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2), \}$

For Exercises 4.a-4.d, Sets A, B, C, and U are defined as follows: $A = \{2, 3, 4, 5\}$; $B = \{4, 5, 6, 7, 8\}$; $C = \{0, 3, 6, 9\}$; $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

4.

   (a) $A \cap B =$

   $A \cap B$ (A intersect B) is the set of all elements that are contained in BOTH sets A and B.

   $A \cap B = \{4, 5\}$

   (b) $\overline{A} =$

   $\overline{A}$ (The complement of A) is the set of all elements in the universe U that are NOT contained in A)

   $\overline{A} = \{0, 1, 6, 7, 8, 9, 10\}$

   (c) $A \cup C =$

   $A \cup C =$ (A union C) is the set of all elements that are contained in EITHER set A or C, or BOTH.

   $A \cup C = \{0, 2, 3, 4, 5, 6, 9\}$

   (d) $B - C =$

   $B - C =$ (the difference of B and C) is the set of all elements of set B that are NOT contained in set C

   $B - C = \{4, 5, 7, 8\}$

5. For arbitrary sets A and B, give an equivalent expression for $\overline{(A \cup B)}$

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ This is one of DeMorgan's Laws for Sets

6. For arbitrary sets A and B, give an equivalent expression for $\overline{(A \cap B)}$

$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ This is the other of DeMorgan's Laws for Sets

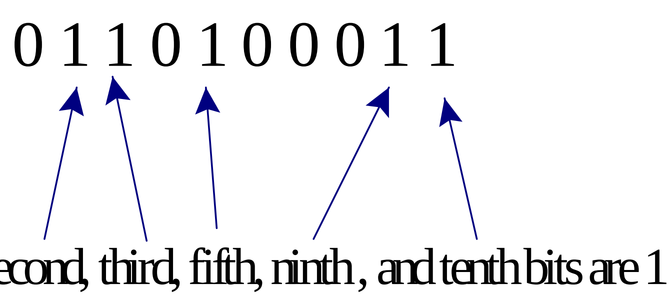
7. Suppose that the Universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   Express the set below with bit strings such that the i^{th} bit is 1 if i is in the set, and the i^{th} bit is 0 otherwise.

   $\{2, 3, 5, 9, 10\}$

   Given a universe U having n elements, the bit string representation of a subset $A \subseteq U$ is the bit string of length n whose i^{th} bit is "1" exactly when the i^{th} element of U is contained in A.

   Since U has ten elements, and since the *second, third, fifth, ninth,* and *tenth* elements of U are in the set $\{2, 3, 5, 9, 10\}$, the bit string representation of $\{2, 3, 5, 9, 10\}$ is the bit string of length 10 whose *second, third, fifth, ninth,* and *tenth* bits are "1"

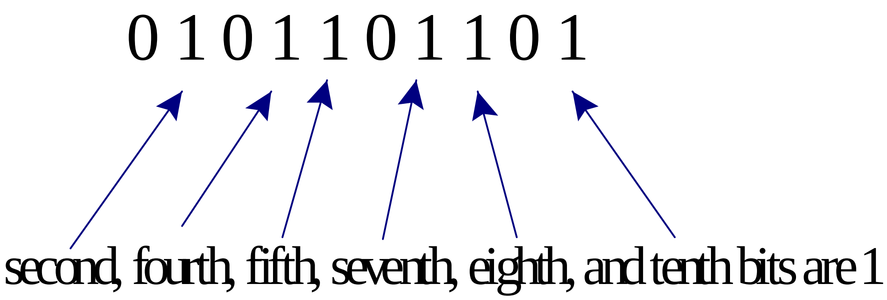


The bit string representation of $\{2, 3, 5, 9, 10\}$ is 0110100011

8. Using the same universal set as in the last problem, find the set specified by the bit string.

   0101101101

   Notice that the *second,fourth,fifth,seventh,eighth,* and *tenth* bits are "1"



   So the bit string 0101101101 represents the subset of U that contains the *second,fourth,fifth, seventh,eighth,* and *tenth* elements of U.

   The bit string 0101101101 represents the set $\{2, 4, 5, 7, 8, 10\}$

9. Compute the following values:

   (a) $\lfloor 2.7 \rfloor$

   This is the *floor function.*

   $\lfloor 2.7 \rfloor$ is the greatest integer less than or equal to 2.7

   The set of integers that are less than or equal to 2.7 are $\{\ldots, -3, -2, -1, 0, 1, 2\}$

   The *greatest* of these is 2.

   Therefore, $\lfloor 2.7 \rfloor = 2$

(b) $\lfloor 5.9 \rfloor$

This is the *floor function.*

$\lfloor 5.9 \rfloor$ is the greatest integer less than or equal to 5.9

The set of integers that are less than or equal to 5.9 are $\{\ldots, -2, -1, 0, 1, 2, 3, 4, 5\}$

The *greatest* of these is 5.

Therefore, $\lfloor 5.9 \rfloor = 5$

(c) $\lfloor 6.0 \rfloor$

This is the *floor function.*

$\lfloor 6.0 \rfloor$ is the greatest integer less than or equal to 6.0

The set of integers that are less than or equal to 6.0 are $\{\ldots, -1, 0, 1, 2, 3, 4, 5, 6\}$

The *greatest* of these is 6.

Therefore, $\lfloor 6.0 \rfloor = 6$

(d) $\lfloor -1.5 \rfloor$

This is the *floor function.*

$\lfloor -1.5 \rfloor$ is the greatest integer less than or equal to -1.5

The set of integers that are less than or equal to -1.5 are $\{\ldots, -6, -5, -4, -3, -2\}$

The *greatest* of these is -2.

Therefore, $\lfloor -1.5 \rfloor = -2$

10. Compute the following values:

(a) $\lceil 8.3 \rceil$

This is the *ceiling function.*

$\lceil 8.3 \rceil$ is the least integer greater than or equal to 8.3

The set of integers that are greater than or equal to 8.3 are $\{9, 10, 11, 12, \ldots\}$

The *least* of these is 9.

Therefore, $\lceil 8.3 \rceil = 9$

(b) $\lceil 4.9 \rceil$

This is the *ceiling function.*

$\lceil 4.9 \rceil$ is the least integer greater than or equal to 4.9

The set of integers that are greater than or equal to 4.9 are $\{5, 6, 7, 8, \ldots\}$

The *least* of these is 5.

Therefore, $\lceil 4.9 \rceil = 5$

(c) $\lceil -6.01 \rceil$

This is the *ceiling function.*

$\lceil -6.01 \rceil$ is the least integer greater than or equal to -6.01

The set of integers that are greater than or equal to -6.01 are $\{-6, -5, -4, -3, \ldots\}$

The *least* of these is -6.

Therefore, $\lceil -6.01 \rceil = -6$

(d) $\lceil -6.99 \rceil$

This is the *ceiling function.*

$\lceil -6.99 \rceil$ is the least integer greater than or equal to -6.99

The set of integers that are greater than or equal to -6.99 are $\{-6, -5, -4, -3, \ldots\}$

The *least* of these is -6.

Therefore, $\lceil -6.99 \rceil = -6$

11. List the first three terms of the sequence whose n^{th} term is given by:

$a_n = 5n - 3$

$a_1 = 5(1) - 3 = 2$

$a_2 = 5(2) - 3 = 7$

$a_3 = 5(3) - 3 = 12$

The first three terms of the sequence are $2, 7, 12$

12. Given the expressions below, [1]write out the terms of the sums and [2]compute the value of the sums

$$\sum_{i=1}^{4}(5i+2) = \underbrace{(5(1)+2)}_{i=1} + \underbrace{(5(2)+2)}_{i=2} + \underbrace{(5(3)+2)}_{i=3} + \underbrace{(5(4)+2)}_{i=4} = 7+12+17+22 = 58$$

(a) i.e., $\sum_{i=1}^{4}(5i+2) = 58$

13. Compute the double sum: $\sum_{i=1}^{2}\sum_{j=1}^{3}(2i+j) =$

Probably the easiest way to do this is to let the index j of the "inner sum" run through all of its values, without assigning the index i a value.

$$\sum_{i=1}^{2}\sum_{j=1}^{3}(2i+j) = \sum_{i=1}^{3}\left(\underbrace{(2i+(1))}_{j=1}+\underbrace{(2i+(2))}_{j=2}+\underbrace{(2i+(3))}_{j=2}\right) = \sum_{i=1}^{3}(6i+6)$$

Then let the index i run through all of its values:

$$\sum_{i=1}^{2}(6i+6) = \underbrace{(6\,(1)+6)}_{i=1}+\underbrace{(6\,(2)+6)}_{i=2} = 12+18 = 30$$

i.e., $\sum_{i=1}^{2}\sum_{j=1}^{3}(2i+j) = 30$

14. Compute the value of the sum $\sum_{i=0}^{7} 3\cdot 2^i$

$$\sum_{i=0}^{7} 3\cdot 2^i = 3+6+12+24+\ldots+384$$

This series is geometric with ratio $r = 2$. (Every term after the first term is equal to 2 times its predecessor)

The sum of this series is given by the formula: $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1}-a}{r-1}$

In the expression: $\sum_{i=0}^{7} 3\cdot 2^i;\quad a = 3;\ r = 2;\ n = 7$

$$\sum_{i=0}^{7} 3\cdot 2^i = \frac{ar^{n+1}-a}{r-1} = \frac{3(2)^{7+1}-3}{2-1} = \frac{3\cdot 256-3}{1} = 765$$

$\sum_{i=0}^{7} 3\cdot 2^i = 765$

15. Find the first six terms of the sequence defined by the recurrence relation: $a_n = -2a_{n-1}$; $a_0 = -1$

$a_0 = -1$

$a_1 = -2a_{1-1} = -2a_0 = -2\,(-1) = 2$

$a_2 = -2a_{2-1} = -2a_1 = -2\,(2) = -4$

$a_3 = -2a_{3-1} = -2a_2 = -2\,(-4) = 8$

$a_4 = -2a_{4-1} = -2a_3 = -2\,(8) = -16$

$a_5 = -2a_{5-1} = -2a_4 = -2\,(-16) = 32$

$$\boxed{\text{The first 6 terms are: } -1, 2, -4, 8, -16, 32}$$