LaPlace transform Homework Problems #3 - Solutions

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Instructions. Solve each initial value problem by using Laplace Transforms. (In each case, assume that y is a function of t.)

1. $y' + 2y = 0;\ \ y(0) = 1$

   (a) Compute the Laplace Transforms of Both Sides of the Equation

   $\pounds[y' + 2y] = \pounds[0]$

   $\Rightarrow \pounds[y'] + 2\pounds[y] = \pounds[0]$

   $\Rightarrow [sY(s) - y(0)] + 2Y(s) = 0$

   (Laplace Transform Table Formulas #16, 19)

   $\Rightarrow [sY(s) - 1] + 2Y(s) = 0$

   (b) Solve Algebraically for $Y(s)$

   $\Rightarrow sY(s) + 2Y(s) = 1$

   $\Rightarrow (s+2)Y(s) = 1$

   $\Rightarrow Y(s) = \frac{1}{s+2}$

   (c) Compute the Inverse Laplace Transform of Both Sides

   $\Rightarrow \pounds^{-1}[Y(s)] = \pounds^{-1}\left[\frac{1}{s+2}\right]$

   $$\boxed{\Rightarrow y = e^{-2t}}$$

   (Inverse Laplace Transform Table - Formula #4)

2. $y'' + 3y' - 4y = 0; \;\; y(0) = 0; \;\; y'(0) = -5$

(a) Compute the Laplace Transforms of Both Sides of the Equation

$\pounds\left[y'' + 3y' - 4y\right] = \pounds\left[0\right]$

$\Rightarrow \pounds\left[y''\right] + 3\pounds\left[y'\right] - 4\pounds\left[y\right] = \pounds\left[0\right]$

$\Rightarrow \left[s^2 Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] - 4Y(s) = 0$

(Laplace Transform Table Formulas #16, 19, 20)

$\Rightarrow \left[s^2 Y(s) - s \cdot 0 - (-5)\right] + 3\left[sY(s) - 0\right] - 4Y(s) = 0$

$\Rightarrow s^2 Y(s) + 5 + 3sY(s) - 4Y(s) = 0$

(b) Solve Algebraically for $Y(s)$

$\Rightarrow s^2 Y(s) + 3sY(s) - 4Y(s) = -5$

$\Rightarrow \left(s^2 + 3s - 4\right) Y(s) = -5$

$\Rightarrow Y(s) = \frac{-5}{s^2+3s-4} = \frac{-5}{(s+4)(s-1)} = \frac{c_1}{s+4} + \frac{c_2}{s-1}$

Scratchwork: ***

$\frac{-5}{(s+4)(s-1)} = \frac{c_1}{s+4} + \frac{c_2}{s-1}$

$\Rightarrow -5 = \frac{c_1}{s+4}(s+4)(s-1) + \frac{c_2}{s-1}(s+4)(s-1) = c_1(s-1) + c_2(s+4)$

i.e., $-5 = c_1(s-1) + c_2(s+4)$

$\boxed{s = 1}$

$\Rightarrow -5 = 5c_2$

$\Rightarrow c_2 = -1$

$\boxed{s = -4}$

$\Rightarrow -5 = -5c_1$

$\Rightarrow c_1 = 1$

End of Scratchwork: ***

Back to our equation:

$Y(s) = \frac{c_1}{s+4} + \frac{c_2}{s-1} = \frac{1}{s+4} + \frac{-1}{s-1}$

i.e. $Y(s) = \frac{1}{s+4} - \frac{1}{s-1}$

(c) Compute the Inverse Laplace Transform of Both Sides

$\Rightarrow \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+4} - \frac{1}{s-1}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+4}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$

$\Rightarrow y = e^{-4t} - e^{t}$

(Inverse Laplace Transform Table - Formula #4)

$$\boxed{\Rightarrow y = e^{-4t} - e^{t}}$$

3. $y'' + 2y' + y = 2te^{-t}; \quad y(0) = 3; \quad y'(0) = -3$

(a) Compute the Laplace Transforms of Both Sides of the Equation

$$\pounds\left[y'' + 2y' + y\right] = \pounds\left[2te^{-t}\right]$$

$$\Rightarrow \pounds\left[y''\right] + 2\pounds\left[y'\right] + \pounds\left[y\right] = 2\pounds\left[te^{-t}\right]$$

$$\Rightarrow \left[s^2Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + Y(s) = \frac{2}{(s-(-1))^2}$$

(Laplace Transform Table Formulas #14, 19, 20)

$$\Rightarrow \left[s^2Y(s) - s\cdot 3 - (-3)\right] + 2\left[sY(s) - 3\right] + Y(s) = \frac{2}{(s+1)^2}$$

$$\Rightarrow s^2Y(s) - 3s + 3 + 2sY(s) - 6 + Y(s) = \frac{2}{(s+1)^2}$$

(b) Solve Algebraically for $Y(s)$

$$\Rightarrow \left(s^2 + 2s + 1\right)Y(s) - 3s - 3 = \frac{2}{(s+1)^2}$$

$$\Rightarrow (s+1)^2\, Y(s) = \frac{2}{(s+1)^2} + 3s + 3$$

$$\Rightarrow Y(s) = \frac{2}{(s+1)^4} + \frac{3s+3}{(s+1)^2} = \frac{2}{(s+1)^4} + \frac{3(s+1)}{(s+1)^2} = \frac{2}{(s+1)^4} + \frac{3}{(s+1)}$$

i.e., $Y(s) = \frac{2}{(s+1)^4} + \frac{3}{(s+1)}$

(c) Compute the Inverse Laplace Transform of Both Sides

$$\Rightarrow \pounds^{-1}\left[Y(s)\right] = \pounds^{-1}\left[\frac{2}{(s+1)^4} + \frac{3}{(s+1)}\right]$$

$$= 2\pounds^{-1}\left[\frac{1}{(s+1)^4}\right] + 3\pounds^{-1}\left[\frac{1}{(s+1)}\right]$$

We want to make these fit Formulas #4 and 5 on the Table of Laplace transform Inverses

$$= 2\pounds^{-1}\left[\frac{1}{3!}\frac{3!}{(s+1)^{3+1}}\right] + 3\pounds^{-1}\left[\frac{1}{(s+1)}\right]$$

$$= \frac{2}{3!}\pounds^{-1}\left[\frac{3!}{(s+1)^{3+1}}\right] + 3\pounds^{-1}\left[\frac{1}{(s+1)}\right]$$

$$= \frac{2}{3!}t^3e^{-t} + 3e^{-t}$$

(Inverse Laplace Transform Table - Formulas #4, 5)

i.e., $y = \frac{1}{3}t^3e^{-t} + 3e^{-t}$

4. $y'' - y = 6e^t; \quad y(0) = 2; \quad y'(0) = 3$

(a) Compute the Laplace Transforms of Both Sides of the Equation

$\pounds[y'' - y] = \pounds[6e^t]$

$\Rightarrow \pounds[y''] - \pounds[y] = 6\pounds[e^t]$

$\Rightarrow [s^2 Y(s) - sy(0) - y'(0)] - Y(s) = 6\frac{1}{(s-1)}$

(Laplace Transform Table Formulas #13, 19, 20)

$\Rightarrow [s^2 Y(s) - s \cdot 2 - 3] - Y(s) = \frac{6}{(s-1)}$

$\Rightarrow s^2 Y(s) - 2s - 3 - Y(s) = \frac{6}{(s-1)}$

(b) Solve Algebraically for $Y(s)$

$\Rightarrow (s^2 - 1) Y(s) - 2s - 3 = \frac{6}{(s-1)}$

$\Rightarrow (s+1)(s-1) Y(s) = \frac{6}{(s-1)} + 2s + 3$

$\Rightarrow Y(s) = \frac{6}{(s-1)^2 (s+1)} + \frac{2s+3}{(s+1)(s-1)}$

Scratchwork: ***

$\frac{6}{(s-1)^2(s+1)} = \frac{c_1}{s-1} + \frac{c_2}{(s-1)^2} + \frac{c_3}{(s+1)}$

$\Rightarrow 6 = c_1(s-1)(s+1) + c_2(s+1) + c_3(s-1)^2$

$\boxed{s = 1}$

$\Rightarrow 6 = 2c_2$

$\Rightarrow c_2 = 3$

$\boxed{s = -1}$

$\Rightarrow 6 = 4c_3$

$\Rightarrow c_3 = \frac{3}{2}$

$\Rightarrow 6 = c_1(s-1)(s+1) + 3(s+1) + \frac{3}{2}(s-1)^2$

$\boxed{s = 0}$

$\Rightarrow 6 = -c_1 + 3 + \frac{3}{2}$

$\Rightarrow \frac{3}{2} = -c_1$

$\Rightarrow c_1 = -\frac{3}{2}$

Hence, $\frac{6}{(s-1)^2(s+1)} = \frac{-\frac{3}{2}}{s-1} + \frac{3}{(s-1)^2} + \frac{\frac{3}{2}}{s+1}$

i.e., $\frac{6}{(s-1)^2(s+1)} = -\frac{3}{2}\frac{1}{s-1} + 3\frac{1}{(s-1)^2} + \frac{3}{2}\frac{1}{(s+1)}$

Also: $\frac{2s+3}{(s+1)(s-1)} = \frac{c_1}{s+1} + \frac{c_2}{s-1}$

$\Rightarrow 2s + 3 = c_1 (s-1) + c_2 (s+1)$

$\boxed{s = 1}$

$\Rightarrow 5 = 2c_2$

$c_2 = \frac{5}{2}$

$\boxed{s = -1}$

$\Rightarrow 1 = -2c_1$

$c_1 = -\frac{1}{2}$

Hence, $\frac{2s+3}{(s+1)(s-1)} = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{5}{2}}{s-1}$

i.e., $\frac{2s+3}{(s+1)(s-1)} = -\frac{1}{2}\frac{1}{s+1} + \frac{5}{2}\frac{1}{s-1}$

End of Scratchwork: **
Back to our equation:

(c) Compute the Inverse Laplace Transform of Both Sides

$$\Rightarrow \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{6}{(s-1)^2(s+1)} + \frac{2s+3}{(s+1)(s-1)}\right]$$

$$= \mathcal{L}^{-1}\left[-\frac{3}{2}\frac{1}{s-1} + 3\frac{1}{(s-1)^2} + \frac{3}{2}\frac{1}{(s+1)} - \frac{1}{2}\frac{1}{s+1} + \frac{5}{2}\frac{1}{s-1}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s-1} + \frac{1}{s+1} + 3\frac{1}{(s-1)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^t + e^{-t} + 3te^{-t}$$

(Inverse Laplace Transform Table - Formulas #4, 5)

$$\boxed{\text{i.e., } y = e^t + e^{-t} + 3te^{-t}}$$

5. $y'' + 10y' + 25y = 2e^{-5t}$; $y(0) = 0$; $y'(0) = -1$

(a) Compute the Laplace Transforms of Both Sides of the Equation

$$\pounds\left[y'' + 10y' + 25y\right] = \pounds\left[2e^{-5t}\right]$$

$$\Rightarrow \pounds\left[y''\right] + 10\pounds\left[y'\right] + 25\pounds\left[y\right] = 2\pounds\left[e^{-5t}\right]$$

$$\Rightarrow \left[s^2 Y(s) - sy(0) - y'(0)\right] + 10\left[sY(s) - y(0)\right] + 25Y(s) = 2\tfrac{1}{(s-(-5))}$$

(Laplace Transform Table Formulas #13, 19, 20)

$$\Rightarrow \left[s^2 Y(s) - s \cdot 0 - (-1)\right] + 10\left[sY(s) - 0\right] + 25Y(s) = \tfrac{2}{(s+5)}$$

$$\Rightarrow s^2 Y(s) + 10sY(s) + 25Y(s) + 1 = \tfrac{2}{(s+5)}$$

(b) Solve Algebraically for $Y(s)$

$$\Rightarrow \left(s^2 + 10s + 25\right) Y(s) + 1 = \tfrac{2}{(s+5)}$$

$$\Rightarrow (s+5)^2 Y(s) = \tfrac{2}{(s+5)} - 1$$

$$\Rightarrow Y(s) = \tfrac{2}{(s+5)^3} - \tfrac{1}{(s+5)^2}$$

(c) Compute the Inverse Laplace Transform of Both Sides

$$\Rightarrow \pounds^{-1}\left[Y(s)\right] = \pounds^{-1}\left[\tfrac{2}{(s+5)^3} - \tfrac{1}{(s+5)^2}\right]$$

$$= \pounds^{-1}\left[\tfrac{2}{(s+5)^3}\right] - \pounds^{-1}\left[\tfrac{1}{(s+5)^2}\right]$$

$$= t^2 e^{-5t} - te^{-5t}$$

(Inverse Laplace Transform Table - Formula #5)

i.e., $y = t^2 e^{-5t} - te^{-5t}$

6. $y'' - 9y' + 18y = 54; \quad y(0) = 0; \quad y'(0) = -3$

(a) Compute the Laplace Transforms of Both Sides of the Equation

$\pounds\left[y'' - 9y' + 18y\right] = \pounds[54]$

$\Rightarrow \pounds[y''] - 9\pounds[y'] + 18\pounds[y] = \pounds[54]$

$\Rightarrow \left[s^2 Y(s) - sy(0) - y'(0)\right] - 9\left[sY(s) - y(0)\right] + 18Y(s) = \frac{54}{s}$

(Laplace Transform Table Formulas #1, 19, 20)

$\Rightarrow \left[s^2 Y(s) - s \cdot 0 - (-3)\right] - 9\left[sY(s) - 0\right] + 18Y(s) = \frac{54}{s}$

$\Rightarrow s^2 Y(s) - 9sY(s) + 18Y(s) + 3 = \frac{54}{s}$

(b) Solve Algebraically for $Y(s)$

$\Rightarrow \left(s^2 - 9s + 18\right) Y(s) + 3 = \frac{54}{s}$

$\Rightarrow (s-6)(s-3)\, Y(s) = \frac{54}{s} - 3$

$\Rightarrow Y(s) = \frac{54}{s(s-6)(s-3)} - \frac{3}{(s-6)(s-3)}$

Scratchwork: **

$\frac{54}{s(s-6)(s-3)} = \frac{c_1}{s} + \frac{c_2}{s-6} + \frac{c_3}{s-3}$

$\Rightarrow 54 = c_1 (s-6)(s-3) + c_2 s (s-3) + c_3 s (s-6)$

$\boxed{s = 0}$

$\Rightarrow 54 = 18c_1$

$\Rightarrow c_1 = 3$

$\boxed{s = 3}$

$\Rightarrow 54 = -9c_3$

$\Rightarrow c_3 = -6$

$\boxed{s = 6}$

$\Rightarrow 54 = 18c_2$

$\Rightarrow c_2 = 3$

Hence, $\frac{54}{s(s-6)(s-3)} = \frac{3}{s} + \frac{3}{s-6} + \frac{-6}{s-3}$

i.e., $\frac{54}{s(s-6)(s-3)} = 3\frac{1}{s} + 3\frac{1}{s-6} - 6\frac{1}{s-3}$

Also: $\frac{3}{(s-6)(s-3)} = \frac{c_1}{s-6} + \frac{c_2}{s-3}$

$\Rightarrow 3 = c_1 (s-3) + c_2 (s-6)$

$\boxed{s = 3}$

$\Rightarrow 3 = -3c_2$

$c_2 = -1$

$\boxed{s = 6}$

$\Rightarrow 3 = 3c_1$

$c_1 = 1$

Hence, $\frac{3}{(s-6)(s-3)} = \frac{1}{s-6} - \frac{1}{s-3}$

End of Scratchwork: **

Back to our equation:

(c) Compute the Inverse Laplace Transform of Both Sides

$$\begin{aligned}
\Rightarrow \mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\frac{54}{s(s-6)(s-3)} - \frac{3}{(s-6)(s-3)}\right] \\
&= \mathcal{L}^{-1}\left[\left(3\frac{1}{s} + 3\frac{1}{s-6} - 6\frac{1}{s-3}\right) - \left(\frac{1}{s-6} - \frac{1}{s-3}\right)\right] \\
&= \mathcal{L}^{-1}\left[3\frac{1}{s} + 2\frac{1}{s-6} - 5\frac{1}{s-3}\right] \\
&= 3\mathcal{L}^{-1}\left[\frac{1}{s}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-6}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] \\
&= 3(1) + 2e^{6t} - 5e^{3t}
\end{aligned}$$

(Inverse Laplace Transform Table - Formulas #1, 5)

$\boxed{\text{i.e., } y = 3 + 2e^{6t} - 5e^{3t}}$

7. $y'' + 9y = e^t; \;\; y(0) = 0; \;\; y'(0) = 0$

(a) Compute the Laplace Transforms of Both Sides of the Equation

$\pounds\left[y'' + 9y\right] = \pounds\left[e^t\right]$

$\Rightarrow \pounds\left[y''\right] + 9\pounds\left[y\right] = \pounds\left[e^t\right]$

$\Rightarrow \left[s^2 Y(s) - sy(0) - y'(0)\right] + 9Y(s) = \frac{1}{(s-1)}$

(Laplace Transform Table Formulas #13, 20)

$\Rightarrow \left[s^2 Y(s) - s \cdot 0 - (0)\right] + 9Y(s) = \frac{1}{(s-1)}$

$\Rightarrow s^2 Y(s) + 9Y(s) = \frac{1}{(s-1)}$

(b) Solve Algebraically for $Y(s)$

$\Rightarrow \left(s^2 + 9\right) Y(s) = \frac{1}{(s-1)}$

$\Rightarrow Y(s) = \frac{1}{(s-1)(s^2+9)}$

Scratchwork: **

$\frac{1}{(s-1)(s^2+9)} = \frac{c_1}{s-1} + \frac{As+B}{s^2+9}$

$\Rightarrow 1 = c_1\left(s^2 + 9\right) + (As + B)(s - 1)$

$\boxed{s = 1}$

$\Rightarrow 1 = 10c_1$

$\Rightarrow c_1 = \frac{1}{10}$

Thus, we have: $1 = \frac{1}{10}\left(s^2 + 9\right) + (As + B)(s - 1)$

$\boxed{s = 0}$

$\Rightarrow 1 = \frac{9}{10} - B$

$\Rightarrow \frac{1}{10} = -B$

$\Rightarrow B = -\frac{1}{10}$

Thus, we have: $1 = \frac{1}{10}\left(s^2 + 9\right) + \left(As - \frac{1}{10}\right)(s - 1)$

$\boxed{s = -1}$

$\Rightarrow 1 = 1 + 2A + \frac{1}{5}$

$\Rightarrow -\frac{1}{5} = 2A$

$\Rightarrow A = -\frac{1}{10}$

Hence, $\frac{1}{(s-1)(s^2+9)} = \frac{c_1}{s-1} + \frac{As+B}{s^2+9} = \frac{\frac{1}{10}}{s-1} + \frac{-\frac{1}{10}s-\frac{1}{10}}{s^2+9}$

i.e., $\frac{1}{(s-1)(s^2+9)} = \frac{1}{10}\left[\frac{1}{s-1} - \frac{s+1}{s^2+9}\right]$

End of Scratchwork: **

Back to our equation:

(c) Compute the Inverse Laplace Transform of Both Sides

$$\Rightarrow \mathcal{L}^{-1}\left[Y\left(s\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s^2+9)}\right] = \mathcal{L}^{-1}\left[\frac{1}{10}\left(\frac{1}{s-1} - \frac{s+1}{s^2+9}\right)\right]$$

$$= \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{10}\mathcal{L}^{-1}\left[\frac{s+1}{s^2+9}\right]$$

$$= \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{10}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] - \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]$$

I want to make $\mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]$ fit formula #6 on the Table of Laplace Transform Inverses, so I have to make the numerator equal to 3

$$= \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{10}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] - \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{3}\frac{3}{s^2+9}\right]$$

$$= \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{10}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] - \frac{1}{30}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{1}{10}e^t - \frac{1}{10}\cos\left(3t\right) - \frac{1}{30}\sin\left(3t\right)$$

(Inverse Laplace Transform Table - Formula #4, 6, 7)

$$\boxed{\text{i.e., } y = \frac{1}{10}e^t - \frac{1}{10}\cos\left(3t\right) - \frac{1}{30}\sin\left(3t\right)}$$

8. $y'' + 10y' + 26y = 37e^t;\quad y(0) = 1;\quad y'(0) = 2$

9. $y''' + y = 1;\quad y(0) = 1;\quad y'(0) = 3;\quad y''(0) = -3$

10. $y'' + 6y' + 9y = 27t;\quad y(0) = 1;\quad y'(0) = 0$

11. $y'' - 3y' - 4y = 25te^{-t};\quad y(0) = 0;\quad y'(0) = 4$

12. $y'' + 2y' - 15y = 16te^{-t} - 15;\quad y(0) = 1;\quad y'(0) = -9$

13. $y'' + 7y' + 10y = 3e^{-2t} - 6e^{-5t};\quad y(0) = 0;\quad y'(0) = 0$

14. $y''' - y = 12\sinh(t);\quad y(0) = 6;\quad y'(0) = -1;\quad y''(0) = 7$

15. $y'' + 4y' + 5y = 39e^t\sin(t);\quad y(0) = -1;\quad y'(0) = -1$

16. $y'' - 4y' + 4y = 3te^{2t} - 4;\quad y(0) = 0;\quad y'(0) = 0$

17. $y''' + 8y = -12e^{-2t};\quad y(0) = -8;\quad y'(0) = 24;\quad y''(0) = -46$

18. $y'' + 7y' + 6y = 250e^t\cos(t);\quad y(0) = 2;\quad y'(0) = -7$