# MTH 3318 - Test #3 - Solutions

SPRING 2024

Pat Rossi  Name ______________________

**Instructions.** Show your work completely. Document your work well.

**Remark 1** *For problems 1 - 3, prove one.*

1. $A \cap B = A \Rightarrow A \subseteq B$

   **Proof.** Let the hypothesis be given. (i.e., let $A \cap B = A$).

   We need to show that $A \subseteq B$.

   So let $x \in A$

   $\Rightarrow x \in A \cap B$ (because $A = A \cap B$ by hypothesis).

   $\Rightarrow x \in A$ and $x \in B$

   In particular, $x \in B$.

   We have just shown that $x \in A \Rightarrow x \in B$.

   Hence, $A \subseteq B$ ■

2. $(A \cup B) = B \Rightarrow A \subseteq B$

   **Proof.** Let the hypothesis be given. (i.e., let $(A \cup B) = B$)

   We need to show that $A \subseteq B$.

   So let $x \in A$

   $\Rightarrow x \in (A \cup B)$ (because $A \subseteq (A \cup B)$ always!)

   $\Rightarrow x \in B$ (because $(A \cup B) = B$ by hypothesis)

   i.e., $x \in A \Rightarrow x \in B$

   Hence, $A \subseteq B$ ■

3. $A \subseteq B \Rightarrow B^c \subseteq A^c$

**Proof.** Let the hypothesis be given. (i.e., let $A \subseteq B$).

We need to show that $B^c \subseteq A^c$.

So let $x \in B^c$

$\Rightarrow x \notin B$

$\Rightarrow x \notin A$ (Otherwise, if $x$ *were* an element of $A$, then our hypothesis would imply that $\Rightarrow x \in B$, contradicting the fact that $x \notin B$.)

$\Rightarrow x \in A^c$.

We have shown that $x \in B^c \Rightarrow x \in A^c$.

Hence, $B^c \subseteq A^c$. ■

**Remark 2** *For problems 4 - 6, prove one.*

4. $(A \cap B)^c = A^c \cup B^c$

   **Proof.** We must show that:

   (a) $(A \cap B)^c \subseteq A^c \cup B^c$

   and

   (b) $A^c \cup B^c \subseteq (A \cap B)^c$

   a. $\boxed{(A \cap B)^c \subseteq A^c \cup B^c}$

   Let $x \in (A \cap B)^c$

   $\Rightarrow x \notin (A \cap B)$

   $\Rightarrow x \notin A$ or $x \notin B$

   $\Rightarrow x \in A^c$ or $x \in B^c$

   $\Rightarrow x \in A^c \cup B^c$

   We have shown that $x \in (A \cap B)^c \Rightarrow x \in A^c \cup B^c$

   Therefore, $(A \cap B)^c \subseteq A^c \cup B^c$

   b. $\boxed{A^c \cup B^c \subseteq (A \cap B)^c}$

   Let $x \in A^c \cup B^c$

   $\Rightarrow x \in A^c$ or $x \in B^c$

   $\Rightarrow x \notin A$ or $x \notin B$

   $\Rightarrow x \notin (A \cap B)$

   $\Rightarrow x \in (A \cap B)^c$

   We have shown that $x \in A^c \cup B^c \Rightarrow x \in (A \cap B)^c$

   Therefore, $A^c \cup B^c \subseteq (A \cap B)^c$ ■

5. $A \subseteq B \Rightarrow (A \cap B) = A$

Let the hypothesis be given. (i.e., let $A \subseteq B$)

We must show that:

a. $\boxed{(A \cap B) \subseteq A}$ (This is *always* true.)

and

b. $\boxed{A \subseteq (A \cap B)}$

Let $x \in A$.

$\Rightarrow x \in B$ (Because $A \subseteq B$, by hypothesis).

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A \cap B$

We have shown that $x \in A \Rightarrow x \in A \cap B$

Therefore, $A \subseteq (A \cap B)$ ■

6. $(A \cup B)^c = A^c \cap B^c$

**Proof.** We must show that:

(a) $(A \cup B)^c \subseteq A^c \cap B^c$

and

(b) $A^c \cap B^c \subseteq (A \cup B)^c$

a. $\boxed{(A \cup B)^c \subseteq A^c \cap B^c}$

Let $x \in (A \cup B)^c$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \in A^c \cap B^c$

i.e., $x \in (A \cup B)^c \Rightarrow x \in A^c \cap B^c$

Thus, $(A \cup B)^c \subseteq A^c \cap B^c$

b. $\boxed{A^c \cap B^c \subseteq (A \cup B)^c}$

Let $x \in A^c \cap B^c$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in (A \cup B)^c$

i.e., $x \in A^c \cap B^c \Rightarrow x \in (A \cup B)^c$

Thus, $A^c \cap B^c \subseteq (A \cup B)^c$ ■

**Remark 3** *Prove problem 7.*

7. $A \cap B = \emptyset \Leftrightarrow (B \cap A^c) = B$

**Proof.** We must show:

(a) $A \cap B = \emptyset \Rightarrow (B \cap A^c) = B$

and

(b) $(B \cap A^c) = B \Rightarrow A \cap B = \emptyset$

a. $\boxed{A \cap B = \emptyset \Rightarrow (B \cap A^c) = B}$

Let the hypothesis be given (i.e., suppose that $A \cap B = \emptyset$)

We must show:

i. $(B \cap A^c) \subseteq B$ (This is *always* true.)

and

ii. $B \subseteq (B \cap A^c)$

Let $x \in B$

$\Rightarrow x \notin A$ (because $A \cap B = \emptyset$ by hypothesis)

$\Rightarrow x \in A^c$

i.e., $x \in B$ and $x \in A^c$

$\Rightarrow x \in B \cap A^c$

i.e., $x \in B \Rightarrow x \in B \cap A^c$

Hence, $B \subseteq (B \cap A^c)$

b. $\boxed{(B \cap A^c) = B \Rightarrow A \cap B = \emptyset}$

Let the hypothesis be given (i.e., suppose that $(B \cap A^c) = B$

To show that $A \cap B = \emptyset$, we must either show that $x \in B \Rightarrow x \notin A$ or that $x \in A \Rightarrow x \notin B$

We will choose the latter: $x \in B \Rightarrow x \notin A$

Let $x \in B$

$\Rightarrow x \in (B \cap A^c)$ (Because $(B \cap A^c) = B$, by our hypothesis)

$\Rightarrow x \in B$ and $x \in A^c$

Specifically, $x \in A^c$

$\Rightarrow x \notin A$

i.e., $x \in B \Rightarrow x \notin A$

Thus, $A \cap B = \emptyset$ ■

**Remark 4** *For problems 8 - 9, prove either one by contradiction.*

8. $(A \cap B) \subseteq A$

   **Proof.** (By contradiction). Suppose, for the sake of deriving a contradiction, that $(A \cap B) \nsubseteq A$

   $\Rightarrow \exists x, \ni x \in (A \cap B)$ and $x \notin A$

   i.e., $\exists x, \ni x \in A$ and $x \in B$, and $x \notin A$

   specifically, $\exists x, \ni x \in A$ and $x \notin A$, which is a contradiction.

   Since the assumption that $(A \cap B) \nsubseteq A$ yields a contradiction, the assumption must be false.

   Hence, $(A \cap B) \subseteq A$ ■

9. $(A \cap B) = \emptyset \Rightarrow A \subseteq B^c$

   **Proof.** (By contradiction). Let the hypothesis, be given.
   i.e., Suppose that $(A \cap B) = \emptyset$.

   Suppose also, for the sake of deriving a contradiction, that $A \nsubseteq B^c$.

   $\Rightarrow \exists x, \ni x \in A$ and $x \notin B^c$

   $\Rightarrow \exists x, \ni x \in A$ and $x \in B$

   $\Rightarrow \exists x, \ni x \in (A \cap B)$, contradicting the assumption that $(A \cap B) = \emptyset$.

   Since the assumption that $(A \cap B) \nsubseteq A$ yields a contradiction, the assumption must be false.

   Hence, $(A \cap B) \subseteq A$ ■

**Remark 5** *For problems 10 - 11, prove either one, by proving the contrapositive.*

10. $\underbrace{A \subseteq B}_{p} \Rightarrow \underbrace{(A \cap B) = A}_{q}$

**Proof.** We will prove the contrapositive, $\underbrace{(A \cap B) \neq A}_{\sim q} \Rightarrow \underbrace{A \nsubseteq B}_{\sim p}$.

Let the hypothesis be given. (i.e., Suppose that $(A \cap B) \neq A$).

$\Rightarrow$ either $(A \cap B) \nsubseteq A$ or $A \nsubseteq (A \cap B)$. (Otherwise, if each set were a subset of the other, the sets would be equal, contrary to our hypothesis.)

Since $(A \cap B) \subseteq A$ (always!) this leaves, as the only possibility, $A \nsubseteq (A \cap B)$.

$\Rightarrow \exists x \in A$ such that $x \notin (A \cap B)$

$\Rightarrow x \in A$, and either: $x \notin A$ or $x \notin B$.

i.e., $\underbrace{x \in A \text{ and } x \notin A}_{\text{impossible}}$, or $x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \notin B$.

i.e., $A$ contains an element that is not contained in $B$.

Hence, $A \nsubseteq B$.

We have shown that $(A \cap B) \neq A \Rightarrow A \nsubseteq B$. ■

11. $\underbrace{(A \cup B) = B}_{p} \Rightarrow \underbrace{A \subseteq B}_{q}$

**Proof.** We will prove the contrapositive, $\underbrace{A \nsubseteq B}_{\sim q} \Rightarrow \underbrace{(A \cup B) \neq B}_{\sim p}$

Let the hypothesis be given. (i.e., Suppose that $A \nsubseteq B$)

We must show that:

a. $(A \cup B) \nsubseteq B$

or

b. $B \nsubseteq (A \cup B)$ (This is impossible because $B \subseteq (A \cup B)$ for ALL sets $A$ and $B$)

Thus, we must show that $(A \cup B) \nsubseteq B$

This just requires that we provide a counter-example.

To create a counter-example, it is often helpful to consider "odd-ball characters" or things that have "unique characteristics."

Let's consider $A = U$, where $U$ is nonempty, and $B = \varnothing$.

Then $(A \cup B) = U \cup \varnothing = U \nsubseteq \varnothing = B$

i.e., $(A \cup B) \nsubseteq B$

We have shown that $\underbrace{A \nsubseteq B}_{\sim q} \Rightarrow \underbrace{(A \cup B) \neq B}_{\sim p}$.

Consequently, $\underbrace{(A \cup B) = B}_{p} \Rightarrow \underbrace{A \subseteq B}_{q}$ ■

**Remark 6** *Disprove problem 12 by providing a counter-example.*

12. $(A \cup B)^c = A^c \cup B^c$

    To create a counter-example, it is often helpful to consider “odd-ball characters” or things that have “unique characteristics.”

    Let’s consider $A = \varnothing$ and $B = U$, where $U$ is nonempty.

    Then $A \cup B = \varnothing \cup U = U$

    $\Rightarrow (A \cup B)^c = U^c = \varnothing$

    i.e., $(A \cup B)^c = \varnothing$

    Also, $A^c = \varnothing^c = U$

    and $B^c = U^c = \varnothing$

    and $A^c \cup B^c = U \cup \varnothing = U$

    i.e., $A^c \cup B^c = U$

    Thus, we have: $(A \cup B)^c = \varnothing \neq U = A^c \cup B^c$

    i.e., $(A \cup B)^c \neq A^c \cup B^c$ ■