MTH 1125 (12 pm) Test #3 Pod B - Solutions

Fall 2020

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 - 3x^2 + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. (Caution - there are **two** critical numbers. Make sure you get them both!)

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 3x^2 - 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 3x^2 - 6x = 0$

   $\Rightarrow 3x(x-2) = 0$

   $\Rightarrow 3x = 0 \quad \text{or} \quad (x-2) = 0$

   $\Rightarrow x = 0$ and $x = 2$ are critical numbers.

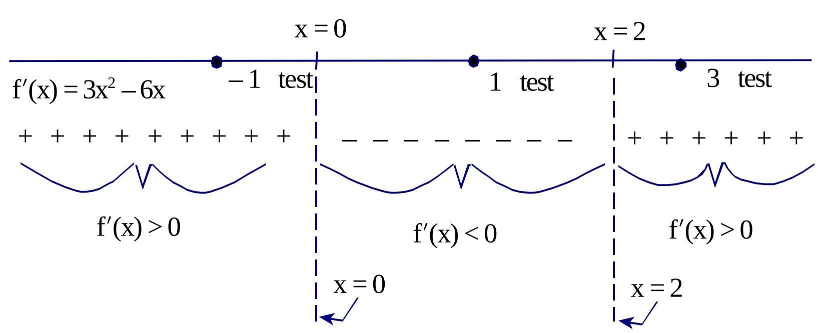
   b. "Type b" $(f'(c)$ is undefined$)$

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



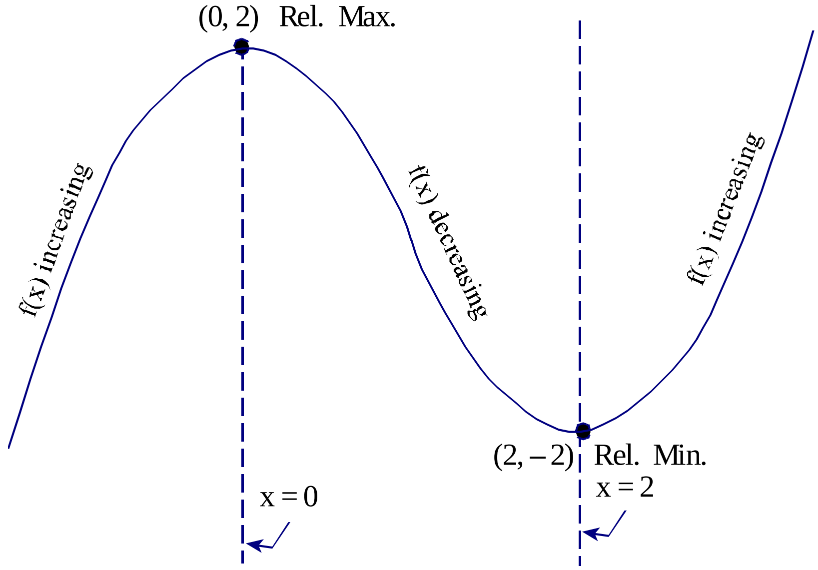
$f(x)$ is **increasing** on the interval(s) $(-\infty, 0)$ and $(2, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(0, 2)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(0, f(0)) = (0, 2)$

Rel Min $(2, f(2)) = (2, -2)$

2. $f(x) = x^4 - 8x^3 - 30x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = 4x^3 - 24x^2 - 60x + 6$

      $f''(x) = 12x^2 - 48x - 60$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 12x^2 - 48x - 60 = 0$

         $\Rightarrow x^2 - 4x - 5 = 0$

         $\Rightarrow (x + 1)(x - 5) = 0$

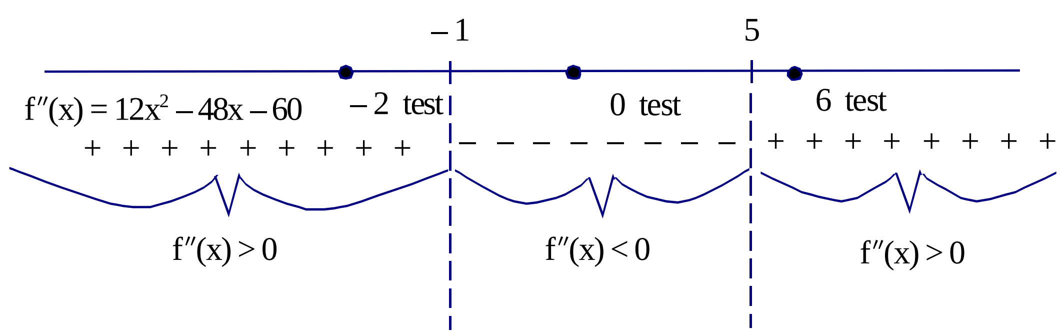
         $x = -1, 5$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined)

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -1)$ and $(5, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-1, 5)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -1$ and $x = 5$, the points:
$(-1, f(-1)) = (-1, -24)$
and
$(5, f(5)) = (5, -1092)$ **are** points of inflection.

3. $f(x) = 2x^3 - 15x^2 - 84x + 3$ on the interval $[-3, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1]$f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-3, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 6x^2 - 30x - 84$

a. "Type a" ($f'(x) = 0$)

$f'(x) = 6x^2 - 30x - 84 = 0$

$\Rightarrow x^2 - 5x - 14 = 0$

$\Rightarrow (x+2)(x-7) = 0$

$\Rightarrow x = -2, 7$ are "type a" critical numbers

Since $x = 7$ is not in the interval $[-3, 2]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-3) = 2(-3)^3 - 15(-3)^2 - 84(-3) + 3 = 66$

$f(-2) = 2(-2)^3 - 15(-2)^2 - 84(-2) + 3 = 95$ ← Abs Max Value

$f(2) = 2(2)^3 - 15(2)^2 - 84(2) + 3 = -209$ ← Abs Min Value

The Abs Max Value is 95 (attained at $x = -2$) The Abs Min Value is -209 (attained at $x = 2$)

4. $f(x) = 2x^{\frac{14}{5}} - 7x^{\frac{4}{5}}$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

$$f'(x) = \frac{28}{5}x^{\frac{9}{5}} - \frac{28}{5}x^{-\frac{1}{5}} = \frac{28x^{\frac{9}{5}}}{5} - \frac{28}{5x^{\frac{1}{5}}} = \frac{28x^{\frac{9}{5}}}{5}\frac{x^{\frac{1}{5}}}{x^{\frac{1}{5}}} - \frac{28}{5x^{\frac{1}{5}}} = \frac{28x^2-28}{5x^{\frac{1}{5}}}$$

i.e., $f'(x) = \frac{28x^2-28}{5x^{\frac{1}{5}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 0$ and solve for x

$\Rightarrow f'(x) = \frac{28x^2-28}{5x^{\frac{1}{5}}} = 0$

$\Rightarrow 28x^2 - 28 = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x+1)(x-1) = 0$

$\Rightarrow x = -1$ and $x = 1$ are critical numbers.

b. "Type b" ($f'(c)$ is undefined)

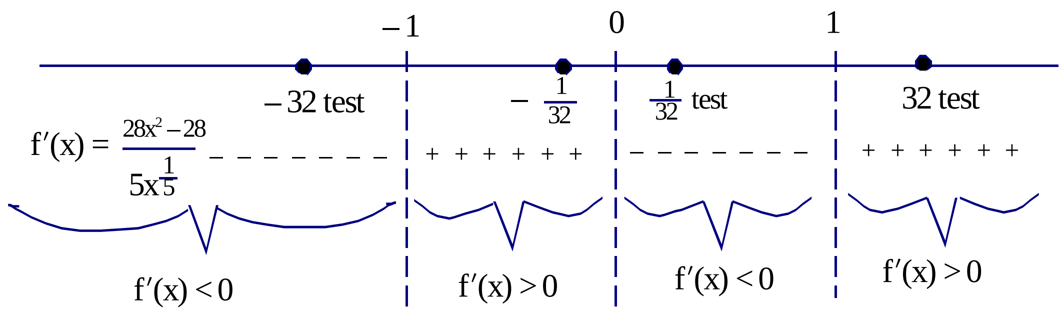
Look for x-value that causes division by zero.

$\Rightarrow 5x^{\frac{1}{5}} = 0$

$\Rightarrow x = 0$ "type b" critical number

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



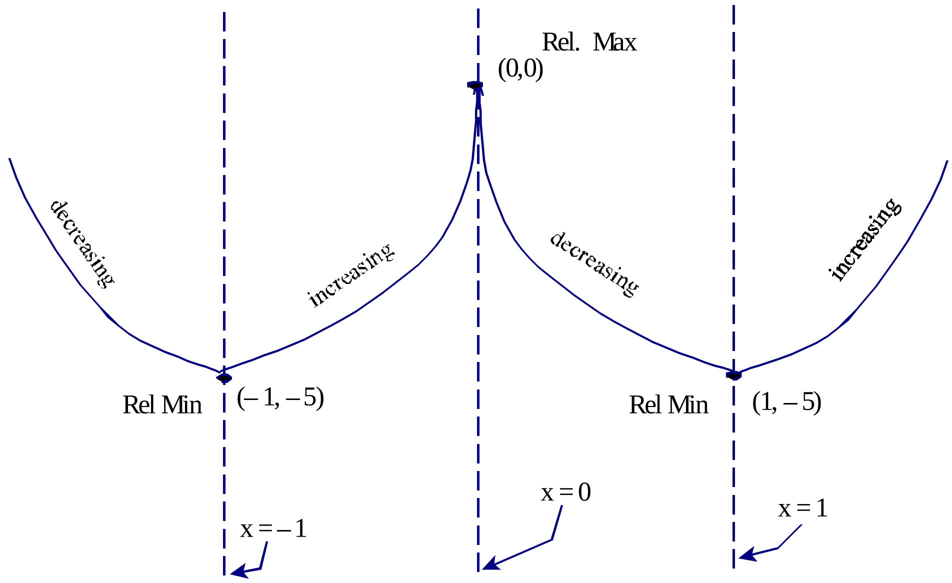
$f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



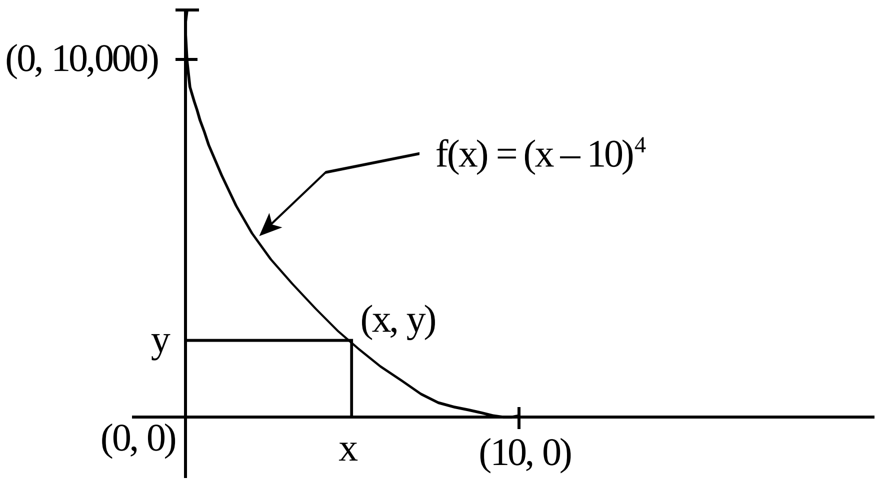
Rel Minimums: $(-1, f(-1)) = (-1, -5)$

and $(1, f(1)) = (1, -5)$

Rel Maximum: $(0, f(0)) = (0, 0)$

5. A rectangle is inscribed in the region bounded by the positive x-axis, the positive y-axis, and the graph of $f(x) = (x - 10)^4$ as shown below. Determine the value of x that makes the area of the rectangle as large as possible.

REMARK: When you create the area function $A(x)$, **do not simplify (i.e. "multiply it out") before you compute the derivative.** If you simplify (i.e. "multiply it out") the area function $A(x)$ before you compute the derivative, it will be very difficult to find the critical numbers.



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $f(x) = (x - 10)^4$.

   Hence, the y-coordinate of the point (x, y) is $y = (x - 10)^4$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(x - 10)^4$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 10$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 10]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = \frac{d}{dx}\left[x(x-10)^4\right] = \underbrace{(x-10)^4 + 4x(x-10)^3}_{\text{Product Rule}}$

      i.e., $A'(x) = (x-10)^4 + 4x(x-10)^3$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = (x-10)^4 + 4x(x-10)^3 = 0$

         $\Rightarrow (x-10)^4 + 4x(x-10)^3 = 0$

         $\Rightarrow (x-10)^3\left[(x-10) + 4x\right] = 0$

         $\Rightarrow (x-10)^3(5x-10) = 0$

         $\Rightarrow x = 10$ and $x = 2$ are critical numbers

      b. "Type b" $(f'(c)$ is undefined)

         Look for x-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = (0)((0)-10)^4 = 0$

      $A(2) = (2)((2)-10)^4 = 8192 \longleftarrow$ Abs Max Value

      $A(10) = (10)((10)-10)^4 = 0$

5. Make sure that we've answered the original question.

   1. "Determine the value of x ..."

      $x = 2$