MTH 3311 Test #1 - Solutions

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Show CLEARLY how you arrive at your answers.

1. Classify the following according to **order** and **linearity.** If an equation is **not linear,** explain why.

   (a) $y'' + x^2 yy' = \sin(x)$ **order 2, non-linear.**

   The highest order of derivative of y is 2. (y'' is the *second derivative* of y.) Since y and y' are factors of the same term, the equation is non-linear.

   (b) $y^{(5)} + x^2 y'' - 2xy = 3x^2 + 2x$ **order 5, linear.**

   The highest order of the derivative of y is 5 ($y^{(5)}$ is the fifth derivative of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

   (c) $e^x y''' - 3xy' + 2x^2 y = \tan(x)$ **order 3, linear.**

   The highest order of the derivative of y is 3 (y''' is the third derivative of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

   (d) $y''' + 2xy'' + \frac{1}{y} = 6x - 6$ **order 3, non-linear.**

   The highest order of derivative of y is 3. (y''' is the *third derivative* of y.) Since y is raised to a power other than 1 ($\frac{1}{y}$ is the same as y^{-1}) the equation is non-linear.

   (e) $y^{(3)} - y'' + 4xy = \frac{x}{x^2+1}$ **order 3, linear.**

   The highest order of the derivative of y is 3. We can rewrite the equation in the form:

$$\underbrace{(1)\, y^{(3)}}_{a_3(x)y'''} + \underbrace{(-1)\, y''}_{a_2(x)y''} + \underbrace{0 \cdot \frac{dy}{dx}}_{a_1(x)y'} + \underbrace{(4x) \cdot y}_{a_0(x)y} = \underbrace{\frac{x}{x^2+1}}_{F(x)}$$

   y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

2. Show that the function $y = c_1e^x + c_2e^{-2x} + 3x^2 + 2x$ is a solution of the differential equation $y'' + y' - 2y = -6x^2 + 2x + 8$

Observe:

$y = c_1e^x + c_2e^{-2x} + 3x^2 + 2x$

$y' = c_1e^x - 2c_2e^{-2x} + 6x + 2$

$y'' = c_1e^x + 4c_2e^{-2x} + 6$

Plugging these into the expression $y'' + y' - 2y$, we have:

$$y'' + y' - 2y = \left(c_1e^x + 4c_2e^{-2x} + 6\right) + \left(c_1e^x - 2c_2e^{-2x} + 6x + 2\right) - 2\left(c_1e^x + c_2e^{-2x} + 3x^2 + 2x\right)$$

$$= (c_1 + c_1 - 2c_1)\,e^x + (4c_2 - 2c_2 - 2c_2)\,e^{-2x} + (6 + 2) + (6 - 4)\,x - 6x^2 = -6x^2 + 2x + 8$$

i.e., $y'' + y' - 2y = -6x^2 + 2x + 8$

Hence, $y = c_1e^x + c_2e^{-2x} + 3x^2 + 2x$ is a solution of the equation:

$y'' + y' - 2y = -6x^2 + 2x + 8$

3. Solve: $\sqrt{x^2-16}\frac{dy}{dx} = \frac{x}{2y}$; subject to the initial condition $y(5) = 2$ (Assume that $x > 4, y > 0$)

Use the "Separation of Variables" Method

$\sqrt{x^2-16}\frac{dy}{dx} = \frac{x}{2y} \Rightarrow 2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}} \Rightarrow 2ydy = \frac{x}{\sqrt{x^2-16}}dx$

i.e., $2ydy = \left(x^2-16\right)^{-\frac{1}{2}} xdx$

The variables are separated, now integrate!

$\int 2ydy = \int \left(x^2-16\right)^{-\frac{1}{2}} xdx$

u	$=$	x^2-16
$\frac{du}{dx}$	$=$	$2x$
du	$=$	$2xdx$
$\frac{1}{2}du$	$=$	xdx

$\Rightarrow \int 2ydy = \int \underbrace{\left(x^2-16\right)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \underbrace{xdx}_{\frac{1}{2}du} = \int u^{-\frac{1}{2}}\frac{1}{2}du = \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C = u^{\frac{1}{2}} + C = \left(x^2-16\right)^{\frac{1}{2}} + C$

i.e., $y^2 = \left(x^2-16\right)^{\frac{1}{2}} + C$

To find C, we refer to the initial condition: $y(5) = 2$

$\Rightarrow (2)^2 = \left((5)^2-16\right)^{\frac{1}{2}} + C$

$\Rightarrow 4 = 9^{\frac{1}{2}} + C$

$\Rightarrow 4 = 3 + C$

$\Rightarrow C = 1$

$\Rightarrow y^2 = \left(x^2-16\right)^{\frac{1}{2}} + 1$

$\Rightarrow \boxed{y^2 - 1 = \left(x^2-16\right)^{\frac{1}{2}}}$

or

$\left(y^2-1\right)^2 = \left(\left(x^2-16\right)^{\frac{1}{2}}\right)^2$

$\Rightarrow y^4 - 2y^2 + 1 = x^2 - 16$

$\Rightarrow \boxed{y^4 - 2y^2 - x^2 = -17}$

4. Solve: $\left(x^2+1\right)\frac{dy}{dx}+3xy=6x$, using the "Integrating Factor" Method

1. Re-write the equation in the form: $\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)$

$$\frac{dy}{dx}+\underbrace{\frac{3x}{x^2+1}}_{P(x)}y=\underbrace{\frac{6x}{x^2+1}}_{Q(x)}$$

2. Compute the integrating factor:

$$e^{\int P(x)dx}=e^{\int\frac{3x}{x^2+1}dx}=e^{\int\frac{1}{x^2+1}3xdx}\ .\ .\ .$$

Scratchwork:

$$\int\underbrace{\frac{1}{x^2+1}}_{\frac{1}{u}}\underbrace{3x}_{\frac{3}{2}du}dx=\int\frac{1}{u}\frac{3}{2}du=\frac{3}{2}\int\frac{1}{u}du=\frac{3}{2}\ln|u|=\frac{3}{2}\ln\left|x^2+1\right|=\frac{3}{2}\ln\left(x^2+1\right)=\ln\left(x^2+1\right)^{\frac{3}{2}}$$

End of Scratchwork

$$e^{\int P(x)dx}=e^{\int\frac{3x}{x^2+1}dx}=e^{\int\frac{1}{x^2+1}3xdx}=e^{\ln\left(x^2+1\right)^{\frac{3}{2}}}=\left(x^2+1\right)^{\frac{3}{2}}$$

3. Multiply both sides by the integrating factor

$$\left(x^2+1\right)^{\frac{3}{2}}\frac{dy}{dx}+\left(x^2+1\right)^{\frac{3}{2}}\frac{3x}{x^2+1}y=\left(x^2+1\right)^{\frac{3}{2}}\frac{6x}{x^2+1}$$

$$\Rightarrow\left(x^2+1\right)^{\frac{3}{2}}\frac{dy}{dx}+\left(x^2+1\right)^{\frac{1}{2}}3xy=\left(x^2+1\right)^{\frac{1}{2}}6x$$

4. Express the left hand side as the derivative of a product

$$\underbrace{\left(x^2+1\right)^{\frac{3}{2}}}_{1^{\text{st}}}\underbrace{\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}}+\underbrace{\frac{3}{2}\left(x^2+1\right)^{\frac{1}{2}}(2x)}_{1^{\text{st}}\text{ prime}}\underbrace{y}_{2^{\text{nd}}}=\left(x^2+1\right)^{\frac{1}{2}}6x$$

$$\Rightarrow\frac{d}{dx}\left[\left(x^2+1\right)^{\frac{3}{2}}y\right]=\left(x^2+1\right)^{\frac{1}{2}}6x$$

5. Integrate both sides w.r.t. x.

$$\int\frac{d}{dx}\left[\left(x^2+1\right)^{\frac{3}{2}}y\right]dx=\int\left(x^2+1\right)^{\frac{1}{2}}6xdx$$

$$\Rightarrow\left(x^2+1\right)^{\frac{3}{2}}y=\int\underbrace{\left(x^2+1\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}}\underbrace{6x}_{3du}dx=\int u^{\frac{1}{2}}3du=3\int u^{\frac{1}{2}}du=3\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C=2u^{\frac{3}{2}}+C=2\left(x^2+1\right)^{\frac{3}{2}}+C$$

i.e., $\left(x^2+1\right)^{\frac{3}{2}}y=2\left(x^2+1\right)^{\frac{3}{2}}+C$

6. Solve for y

$$\Rightarrow y=2+C\left(x^2+1\right)^{-\frac{3}{2}}$$

Our solution is $y=2+C\left(x^2+1\right)^{-\frac{3}{2}}$

5. Determine whether or not the equation is exact. If the equation is exact, solve it.

$$\left(2e^{2x} + 15x^2y^2\right) dx + \left(10x^3y + \sin(y)\right) dy = 0$$

This equation can be analyzed as:

$$\underbrace{\left(2e^{2x} + 15x^2y^2\right)}_{M(x,y)} dx + \underbrace{\left(10x^3y + \sin(y)\right)}_{N(x,y)} dy = 0$$

By convention, we let $M(x,y)$ be the co-factor of dx and we let $N(x,y)$ be the co-factor of dy.

i.e., $M(x,y) = 2e^{2x} + 15x^2y^2$ and $N(x,y) = 10x^3y + \sin(y)$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 30x^2y = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x,y)$ such that the equation $U(x,y) = C$ relates the solution y implicitly as a function of x.

To find $U(x,y)$, we compute the integrals $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$$U(x,y) = \int M(x,y)\,dx = \int \left(2e^{2x} + 15x^2y^2\right) dx = e^{2x} + 5x^3y^2 + f(y) + C$$

$$U(x,y) = \int N(x,y)\,dy = \int \left(10x^3y + \sin(y)\right) dy = 5x^3y^2 - \cos(y) + g(x) + C$$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$$\begin{array}{ccccccccc} U(x,y) & = & 5x^3y^2 & + & e^{2x} & + & (-\cos(y)) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \\ U(x,y) & = & 5x^3y^2 & + & g(x) & + & f(y) & + & C \end{array}$$

Thus, $f(y) = -\cos(y)$ and $g(x) = e^{2x}$, and $U(x,y) = 5x^3y^2 + e^{2x} - \cos(y) + C$

Our solution $y = y(x)$ is given implicitly by the equation $U(x,y) = C$

$$\boxed{5x^3y^2 + e^{2x} - \cos(y) = C}$$

6. Solve: $2xy\frac{dy}{dx} = 4x^2 + 3y^2$ using the substitution $v = \frac{y}{x}$. (Assume that $x, y > 0$)

Re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$2xy\frac{dy}{dx} = 4x^2 + 3y^2$

$\Rightarrow \frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy}$

$\Rightarrow \frac{dy}{dx} = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = 2\frac{1}{\left(\frac{y}{x}\right)} + \frac{3}{2}\left(\frac{y}{x}\right)$ (Eq. 1)

Thus, we have re-expressed the equation in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting into Eq. 1, we have:

$v + x\frac{dv}{dx} = 2\frac{1}{v} + \frac{3}{2}v$ Now Separate!

$\Rightarrow x\frac{dv}{dx} = 2\frac{1}{v} + \frac{3}{2}v - v = \frac{2}{v} + \frac{v}{2} = \frac{4}{2v} + \frac{v^2}{2v} = \frac{v^2+4}{2v}$

i.e., $x\frac{dv}{dx} = \frac{v^2+4}{2v}$

$\Rightarrow \frac{2v}{v^2+4}dv = \frac{1}{x}dx$

Now Integrate!

$\Rightarrow \int \frac{2v}{v^2+4}dv = \int \frac{1}{x}dx \Rightarrow \ln\left(v^2+4\right) = \ln(x) + C \Rightarrow e^{\ln\left(v^2+4\right)} = e^{\ln(x)+C}$

$\Rightarrow v^2 + 4 = e^{\ln(x)}e^C \Rightarrow v^2 + 4 = C_1 x$

Substituting $\frac{y}{x}$ for v, we have:

$\Rightarrow \left(\frac{y}{x}\right)^2 + 4 = C_1 x \Rightarrow \frac{y^2}{x^2} + 4 = C_1 x \Rightarrow y^2 + 4x^2 = C_1 x^3$

Our solution y is given implicitly by the equation:

$$\boxed{y^2 + 4x^2 = C_1 x^3}$$