Definitions, Theorems, Proofs to Know for Test #2

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Proofs to Know for the Test

Prove or Disprove: $(\mathbb{Q}, +)$ is a cyclic group

This is False.

pf/ Suppose, for the sake of deriving a contradiction, that $(\mathbb{Q}, +)$ IS cyclic.

Then $\exists\ a, b \in \mathbb{Z}$ such that $\left\langle \frac{a}{b} \right\rangle = (\mathbb{Q}, +)$

This means that every positive rational number must be of the form $n\left(\frac{a}{b}\right)$, for some $n \in \mathbb{Z}$.

What about the rational number $\frac{a}{2b}$?

Since $\frac{a}{b}$ generates $\mathbb{Q}$, $\frac{a}{2b} = n\left(\frac{a}{b}\right)$, for some $n \in \mathbb{Z}$.

But $n\left(\frac{a}{b}\right) = \frac{a}{2b} \Rightarrow n = \frac{1}{2}$. contradicting the fact that $n \in \mathbb{Z}$.

Since this contradiction is a consequence of our assumption that $(\mathbb{Q}, +)$ is cyclic, the assumption must be false.

Hence, $(\mathbb{Q}, +)$ is NOT cyclic ■

Prove or Disprove: $(\mathbb{R}, +)$ is a cyclic group

This is False.

pf/ If $(\mathbb{R}, +)$ were a cyclic group, then $(\mathbb{Q}, +)$ would be cyclic also, since every subgroup of a cyclic group is cyclic also.

But $(\mathbb{Q}, +)$ is NOT cyclic.

Hence, $(\mathbb{R}, +)$ is not cyclic. ■

Prove: $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$

pf/ We claim that: $f : \mathbb{R} \to \mathbb{R}^+$, given by $f(x) = e^x$ is our isomorphism.

Note that $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, given by $f^{-1}(x) = \ln(x)$, is the inverse of f

$(f \circ f^{-1}) : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $(f \circ f^{-1}) = 1_{\mathbb{R}^+}$

and

$(f^{-1} \circ f) : \mathbb{R} \to \mathbb{R}$ is such that $(f^{-1} \circ f) = 1_{\mathbb{R}}$

Hence, $f : \mathbb{R} \to \mathbb{R}^+$, given by $f(x) = e^x$ is one to one and onto.

To show that f is an isomorphism and that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ isomorphic, we need to show that: $f(r_1 + r_2) = f(r_1) \cdot f(r_2)$

Observe: $f(r_1 + r_2) = e^{r_1 + r_2} = e^{r_1} \cdot e^{r_2} = f(r_1) \cdot f(r_2)$

Hence, $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic ■

Ex Show that $(\mathbb{R}^+, \cdot) \cong (\mathbb{R}, +)$ by showing that $g : \mathbb{R}^+ \to \mathbb{R}$, given by $g(x) = \ln(x)$ is an isomorphism.

Note that $g^{-1} : \mathbb{R} \to \mathbb{R}^+$, given by $g^{-1}(x) = e^x$, is the inverse of g

$(g \circ g^{-1}) : \mathbb{R} \to \mathbb{R}$ is such that $(g \circ g^{-1}) = 1_{\mathbb{R}}$

and

$(g^{-1} \circ g) : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $(g^{-1} \circ g) = 1_{\mathbb{R}^+}$

Hence, $g : \mathbb{R}^+ \to \mathbb{R}$, given by $g(x) = \ln(x)$ is one to one and onto.

To show that f is an isomorphism and that $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ isomorphic, we need to show that:

$g(r_1 \cdot r_2) = g(r_1) + g(r_2)$

Observe: $g(r_1 \cdot r_2) = \ln(r_1 \cdot r_2) = \ln(r_1) + \ln(r_2) = g(r_1) \cdot g(r_2)$

Hence, $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic ■

Exercises to Work on in Preparation for the Final Exam

Go to my Website:
www.pat-rossi.com
Academic Links . . .
MTH 4441
Study the following:

Homework Assignment #5 – #1-13, ~~17 only~~ 15- 17

Homework Assignment #6 – ALL

Homework Assignment #7 – #1-14 only