MTH 4436 HW Set 2.1

Summer 2023

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Set 2.1

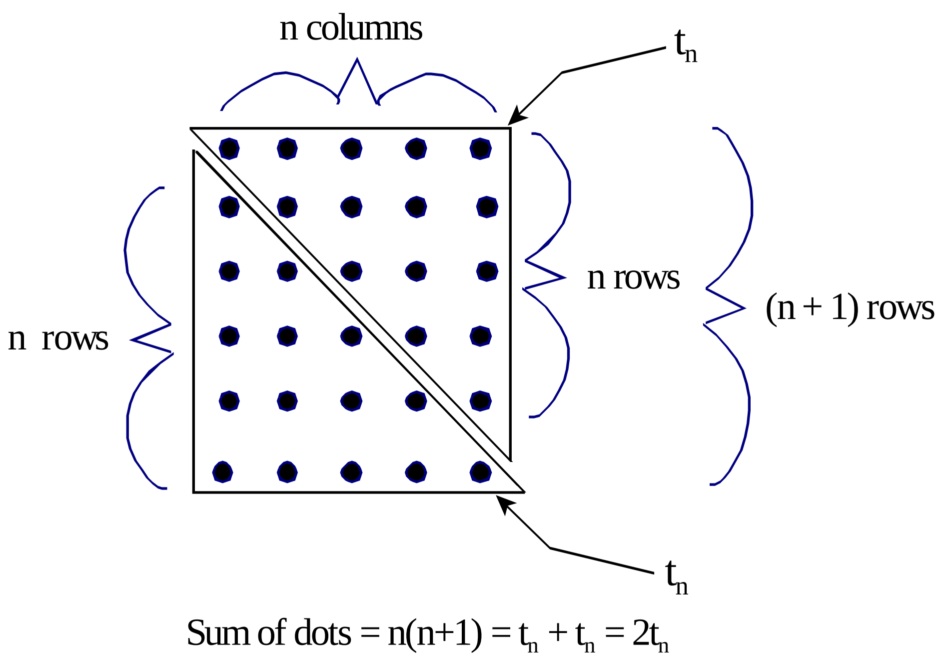
1.a. A number is triangular if and only if it is of the form $\frac{n(n+1)}{2}$.

Recall that the n^{th} triangular number t_n is equal to the number of dots in a right, equilateral triangle that consists of n rows of dots. The first row has 1 dot and each row after that has one more dot than its predecessor. This is tantamount to saying that $\forall n \in \mathbb{N}$, $t_n = 1 + 2 + 3 + \ldots + n$.

In Exercise 1.a on page 5 of our text, we proved (by induction) that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

Thus, $\forall n \in \mathbb{N}$, $t_n = \frac{n(n+1)}{2}$

Alternatively: We can use a "dot diagram proof."



1. Observe that by placing 2 copies of the dots corresponding to the n^{th} triangular number t_n, as shown above, we create a rectangle of dots having $(n+1)$ rows and n columns of dots. (i.e., having $n(n+1)$ total dots.)

   Thus, $t_n + t_n = n(n+1)$.

   or: $2t_n = n(n+1)$.

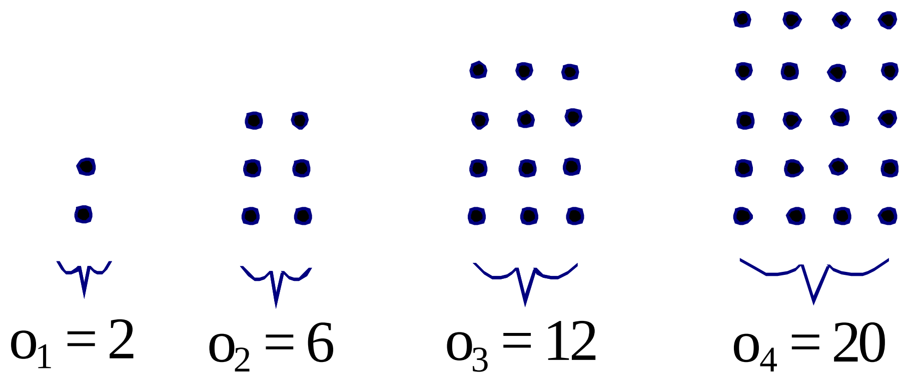
   Hence, $t_n = \frac{n(n+1)}{2}$.

1.b. The integer n is a triangular number if and only if $8n + 1$ is a perfect square.

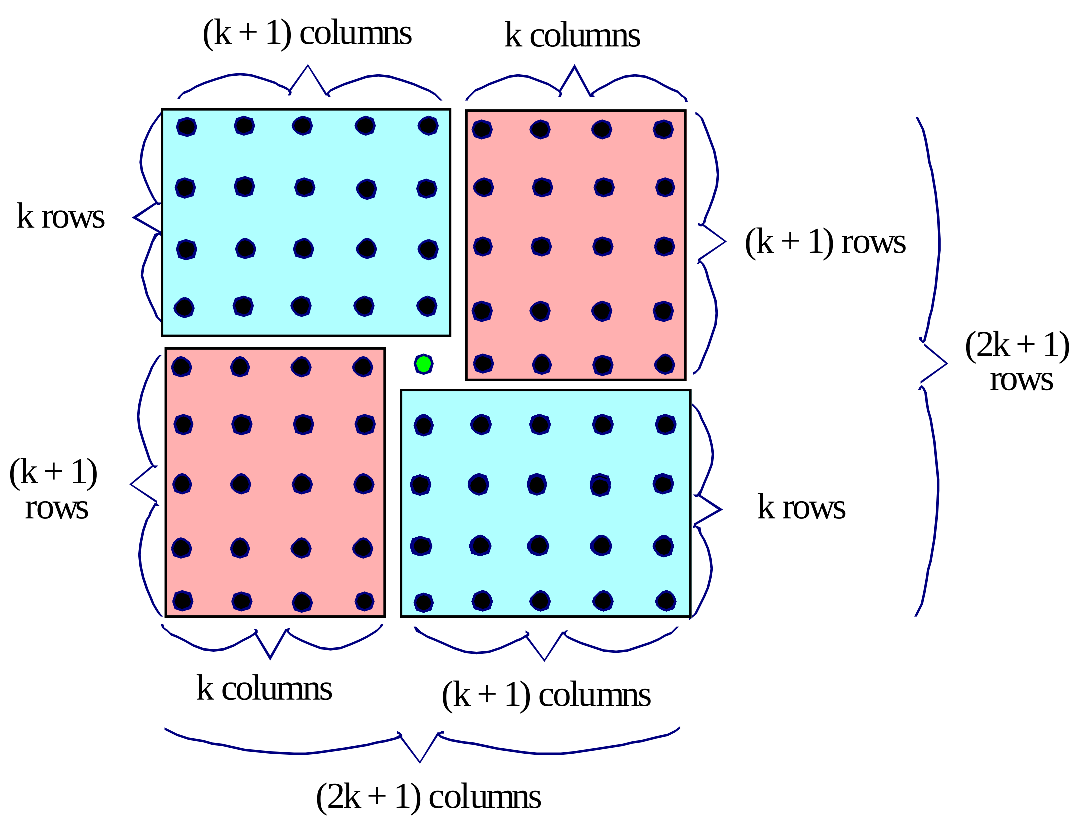
This one's not obvious (at least it wasn't obvious to me!)

To make a transition form the mysterious to the obvious, I am going to prove an "intermediate result," Namely, that the integer n is an **oblong** number if and only if $4n + 1$ is a perfect square.

Recall: The n^{th} oblong number o_n is the number of dots in a rectangular array of dots in which there are $n + 1$ rows and n columns. The first few oblong numbers are shown below:



Now note that 4 rectangles, each containing o_k number of dots can be positioned around a single dot so as to form a square containing $(2k + 1)$ rows and $(2k + 1)$ columns of dots.

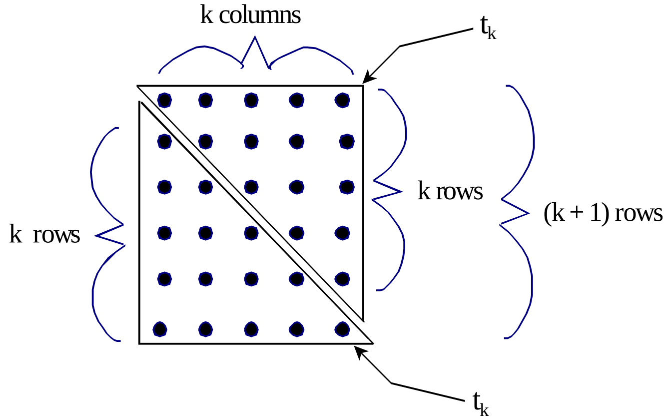


Sum of dots $= 4\,o_k + 1 = (2k + 1)^2$

We have established that if n is an oblong number, then $4n+1$ is a perfect square.

Our next step is to show that for any $n \in \mathbb{N}$, $t_n + t_n = o_n$ (i.e., $2t_n = o_n$).

This can be seen from the fiagram below:

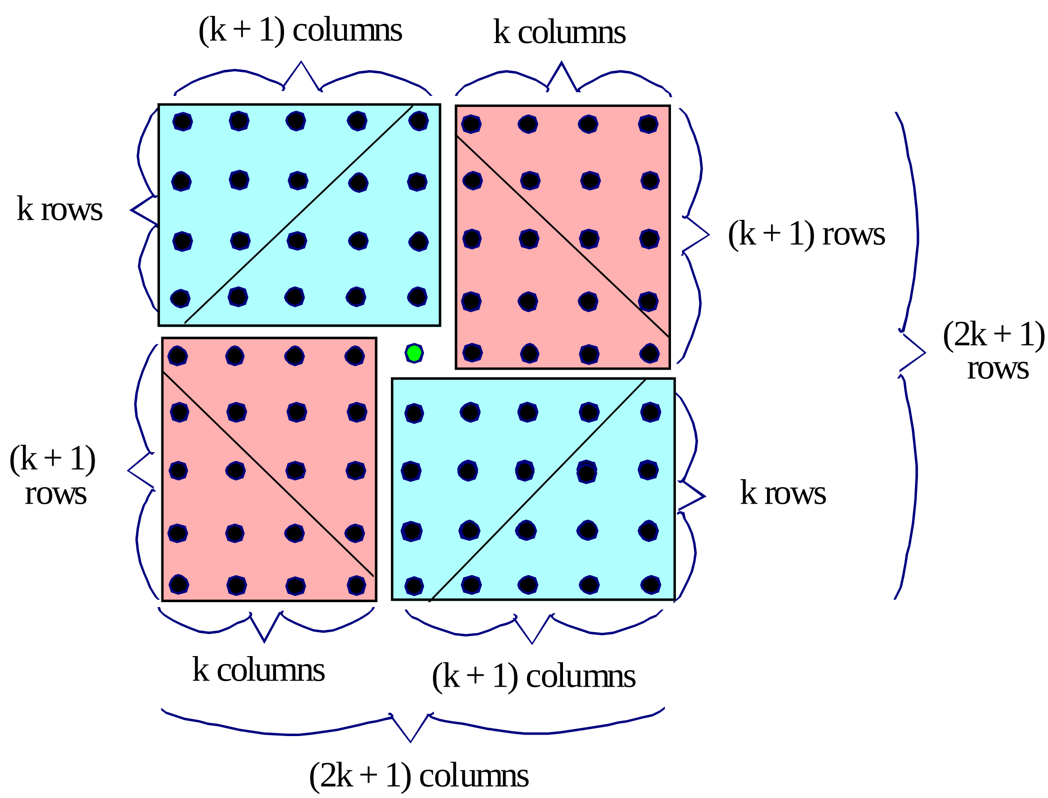


Sum of dots = $o_k = 2t_k$

Thus, for any $n \in \mathbb{N}$, $(2n+1)^2 = 4o_n + 1 = 4(2t_n) + 1 = 8t_n + 1$

i.e., $8t_n + 1 = (2n+1)^2$, for any $n \in \mathbb{N}$.

This fact is depicted graphically, below:



Sum of dots $= 8\,t_k + 1 = (2k+1)^2$

1.c. The sum, of any two consecutive triangular numbers is a perfect square.

Note that $t_{n+1} = t_n + (n+1)$. (Geometrically, the $n+1^{st}$ triangle is formed by taking the n^{th} triangle and adding a row containing $n+1$ dots.)

Combine this with the result from part a) that the n^{th} triangular number, $t_n = \frac{n(n+1)}{2}$, and we have:

$$t_n + t_{n+1} = t_n + [t_n + (n+1)] = 2t_n + n + 1 = 2\frac{n(n+1)}{2} + n + 1 = n(n+1) + n + 1$$

$$= (n+1)(n+1) = (n+1)^2$$

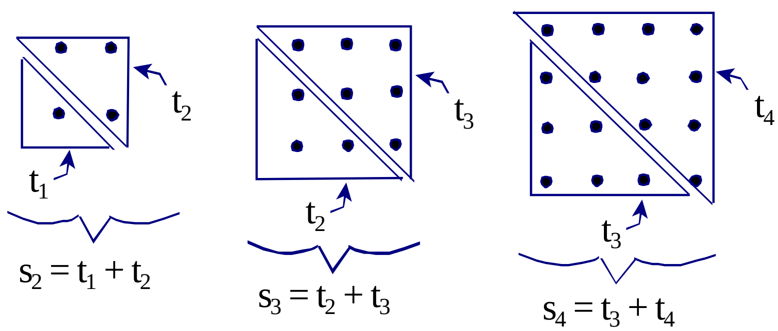
i.e., $t_n + t_{n+1} = (n+1)^2$

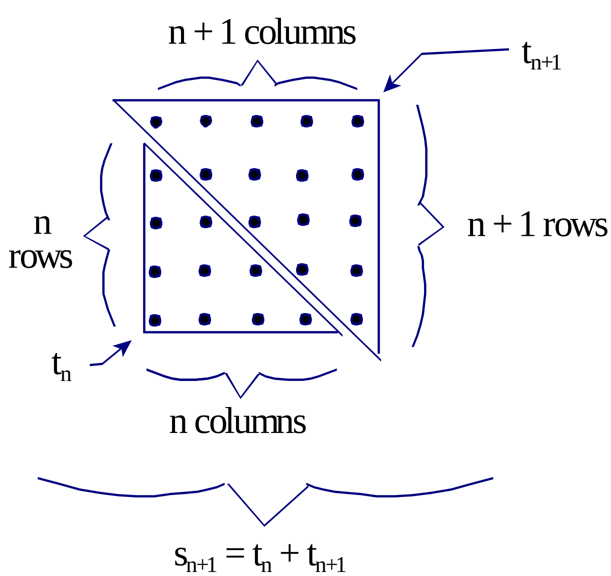
Hence, the sum of any two consecutive triangular numbers is a perfect square.

Alternatively: We can use a "dot diagram proof."

Observe, from the dot diagrams below, that two "dot diagrams" - one representing the n^{th} triangular number t_n, and one representing the $(n+1)^{\text{st}}$ triangular number t_{n+1}, can be situated in such a way as to form a square containing $(n+1)$ rows and columns of dots.

Thus, $t_n + t_{n+1} = (n+1)^2$.





7. Show that the difference between the squares of two consecutive triangular numbers is always a cube.

Proof:

Recall: $t_n = \frac{n(n+1)}{2}$ is the n$^{\text{th}}$ triangular number.

Consequently: $t_{n+1} = \frac{(n+1)(n+2)}{2}$

Consider: $t_{n+1}^2 - t_n^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2$

$$= \underbrace{\left(\frac{(n+1)(n+2)}{2} + \frac{n(n+1)}{2}\right)\left(\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2}\right)}_{\text{Difference of Perfect Squares}}$$

$$= \left(\frac{(n+1)[(n+2)+n]}{2}\right)\left(\frac{(n+1)[(n+2)-n]}{2}\right) = \left(\frac{(n+1)(2n+2)}{2}\right)\left(\frac{(n+1)(2)}{2}\right)$$

$$= \left(\frac{(n+1)(n+1)(2)}{2}\right)\left(\frac{(n+1)(2)}{2}\right) = (n+1)(n+1)(n+1) = (n+1)^3$$

i.e., $t_{n+1}^2 - t_n^2 = s_{n+1}^3$