

Problem Set 2.4; page 31

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Name _____

1. Find the $\gcd(143, 227)$, $\gcd(306, 657)$, $\gcd(272, 1479)$.

We find all of these by using the Division Algorithm.

(a) $\gcd(143, 227)$

$$227 = q_1(143) + r_1$$

$$227 = (1)(143) + 84$$

Repeat with 143 and 84.

$$143 = q_2(84) + r_2$$

$$143 = (1)(84) + 59$$

Repeat with 84 and 59.

$$84 = q_3(59) + r_3$$

$$84 = (1)(59) + 25$$

Repeat with 59 and 25.

$$59 = q_4(25) + r_4$$

$$59 = (2)(25) + 9$$

Repeat with 25 and 9.

$$25 = q_5(9) + r_5$$

$$25 = (2)(9) + 7$$

$\gcd(143, 227) =$ last non-zero remainder

$$\gcd(143, 227) = 1$$

(b) $\gcd(306, 657)$

$$657 = q_1(306) + r_1$$

$$657 = (2)(306) + 45$$

Repeat with 306 and 45

$$306 = q_2(45) + r_2$$

$$306 = (6)(45) + 36$$

Repeat with 45 and 36

$$45 = q_3(36) + r_3$$

$$45 = (1)(36) + 9$$

Repeat with 36 and 9.

$$36 = q_4(9) + r_4$$

$$36 = (4)(9) + 0$$

$\gcd(306, 657) =$ the last non-zero remainder.

$$\gcd(306, 657) = 9$$

(c) $\gcd(272, 1479)$

$$1479 = q_1(272) + r_1$$

$$1479 = (5)(272) + 119$$

Repeat with 272 and 119.

$$272 = q_2(119) + r_1$$

$$272 = (2)(119) + 34$$

Repeat with 119 and 34.

$$119 = q_3(34) + r_3$$

$$119 = (3)(34) + 17$$

Repeat with 34 and 17.

$$34 = q_4(17) + r_4$$

$$34 = (2)(17) + 0$$

$\gcd(272, 1479) = \text{last non-zero remainder}$

$$\gcd(272, 1479) = 17$$

2. Use the Euclidean Algorithm to obtain integers x and y satisfying the following:

(a) $\gcd(56, 72) = 56x + 72y$

We find $\gcd(56, 72)$ by using the Euclidean Algorithm, and then “retracing our steps.”

$$72 = q_1(56) + r_1$$

$$72 = (1)(56) + 16 \quad (\text{eq. 2})$$

Repeat using 56 and 16

$$56 = q_2(16) + r_2$$

$$56 = (3)(16) + 8 \quad (\text{eq. 1})$$

Repeat using 16 and 8.

$$16 = q_3(8) + r_3$$

$$16 = (2)(8) + 0$$

$\gcd(56, 72) =$ last non-zero remainder

$$\gcd(56, 72) = 8$$

So, we want x and y such that $56x + 72y = 8$

$$8 = 56 - (3)(16) \quad (\text{From eq. 1})$$

$$16 = 72 - (1)(56) \quad (\text{From eq.2})$$

$$\Rightarrow 8 = 56 - (3)(72 - (1)(56))$$

$$\Rightarrow 8 = (4)(56) - (3)(72)$$

$$\text{i.e., } 56(4) + 72(-3) = 8$$

Our particular solution is $(x_0, y_0) = (4, -3)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$; for $t \in \mathbb{Z}$

$$= \left(\frac{72}{8}t, -\frac{56}{8}t\right) = (9t, -7t); \text{ for } t \in \mathbb{Z}$$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (4, -3) + (9t, -7t) = (4 + 9t, -3 - 7t); \text{ for } t \in \mathbf{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (4 + 9t, -3 - 7t); \text{ for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 4 + 9t; \quad y = -3 - 7t \quad \text{for } t \in \mathbf{Z}$$

(b) $\gcd(24, 138) = 24x + 138y$

We find $\gcd(24, 138)$ by using the Euclidean Algorithm, and then “retracing our steps.”

$$138 = q_1(24) + r_1$$

$$138 = (5)(24) + 18 \quad (\text{eq. 2})$$

Repeat using 24 and 18.

$$24 = q_2(18) + r_2$$

$$24 = (1)(18) + 6 \quad (\text{eq. 1})$$

Repeat using 18 and 6.

$$18 = q_3(6) + r_3$$

$$18 = (3)(6) + 0$$

$\gcd(24, 138) =$ last non-zero remainder

$$\gcd(24, 138) = 6$$

So, we want x and y such that $24x + 138y = 6$

$$6 = 24 - (1)(18) \quad (\text{From eq. 1})$$

$$18 = 138 - (5)(24) \quad (\text{From eq. 2})$$

$$\Rightarrow 6 = 24 - (1)(138 - (5)(24))$$

$$\Rightarrow 6 = (6)(24) - (1)(138)$$

$$\text{i.e., } 24(6) + 138(-1) = 6$$

Our particular solution is $(x_0, y_0) = (6, -1)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right); \quad \text{for } t \in \mathbb{Z}$

$$= \left(\frac{138}{6}t, -\frac{24}{6}t\right) = (23t, -4t); \quad \text{for } t \in \mathbb{Z}$$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (6, -1) + (23t, -4t) = (6 + 23t, -1 - 4t); \quad \text{for } t \in \mathbb{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (6 + 23t, -1 - 4t); \quad \text{for } t \in \mathbb{Z}$$

$$\text{i.e., } x = 6 + 23t; \quad y = -1 - 4t \quad \text{for } t \in \mathbb{Z}$$

(c) $\gcd(119, 272) = 119x + 272y$

We find $\gcd(119, 272)$ by using the Euclidean Algorithm, and then “retracing our steps.”

$$272 = q_1 119 + r_1$$

$$272 = (2)(119) + 34 \quad (\text{eq. 2})$$

Repeat for 119 and 34

$$119 = q_2(34) + r_2$$

$$119 = (3)(34) + 17 \quad (\text{eq. 1})$$

Repeat for 34 and 17

$$34 = q_3(17) + r_3$$

$$34 = (2)(17) + 0$$

$\gcd(119, 272) =$ last non-zero remainder

$$\gcd(119, 272) = 17$$

So, we want x and y such that $119x + 272y = 17$

$$17 = 119 - (3)(34) \quad (\text{From eq. 1})$$

$$34 = 272 - (2)(119) \quad (\text{From eq. 2})$$

$$\Rightarrow 17 = 119 - (3)(272 - (2)(119))$$

$$\Rightarrow 17 = (7)(119) - (3)(272)$$

$$\text{i.e., } 119(7) + 272(-3) = 17$$

Our particular solution is $(x_0, y_0) = (7, -3)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right); \quad \text{for } t \in \mathbb{Z}$

$$= \left(\frac{272}{17}t, -\frac{119}{17}t\right) = (16t, -7t); \quad \text{for } t \in \mathbb{Z}$$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (7, -3) + (16t, -7t) = (7 + 16t, -3 - 7t); \quad \text{for } t \in \mathbb{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (7 + 16t, -3 - 7t); \quad \text{for } t \in \mathbb{Z}$$

$$\text{i.e., } x = 7 + 16t; \quad y = -3 - 7t \quad \text{for } t \in \mathbb{Z}$$

(d) $\gcd(1769, 2378) = 1769x + 2378y$

Find $\gcd(1769, 2378)$ by using the Euclidean Algorithm and retracing steps.

$$2378 = q_1(1769) + r_1$$

$$2378 = (1)(1769) + 609 \tag{eq. 4}$$

Repeat for 1769 and 609

$$1769 = q_2(609) + r_2$$

$$1769 = (2)(609) + 551 \tag{eq. 3}$$

Repeat with 609 and 551

$$609 = q_3(551) + r_3$$

$$609 = (1)(551) + 58 \tag{eq. 2}$$

Repeat with 551 and 58

$$551 = q_4(58) + r_4$$

$$551 = (9)(58) + 29 \tag{eq. 1}$$

Repeat with 58 and 29

$$58 = q_5(29) + r_5$$

$$58 = (2)(29) + 0$$

$\gcd(1769, 2378) =$ last non-zero remainder

$$\gcd(1769, 2378) = 29$$

So, we want x and y such that $1769x + 2378y = 29$

$$29 = 551 - (9)(58) \quad (\text{From eq. 1})$$

$$58 = 609 - (1)(551) \quad (\text{From eq. 2})$$

$$\Rightarrow 29 = 551 - (9)(609 - (1)(551))$$

$$\Rightarrow 29 = (-9)(609) + (10)(551)$$

$$551 = 1769 - (2)(609) \quad (\text{From eq. 3})$$

$$\Rightarrow 29 = (-9)(609) + (10)(1769 - (2)(609))$$

$$\Rightarrow 29 = (10)(1769) + (-29)(609)$$

$$609 = 2378 - (1)(1769) \quad (\text{From eq. 4})$$

$$\Rightarrow 29 = (10)(1769) + (-29)(2378 - (1)(1769))$$

$$\Rightarrow 29 = (39)(1769) + (-29)(2378)$$

$$\text{i.e., } 1769(39) + 2378(-29) = 29$$

Our particular solution is $(x_0, y_0) = (39, -29)$

The homogeneous solution is $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$; for $t \in \mathbb{Z}$
 $= \left(\frac{2378}{29}t, -\frac{1769}{29}t\right) = (82t, -61t)$; for $t \in \mathbb{Z}$

The general solution is the sum of the particular solution and the homogeneous solution.

$$(x, y) = (x_0, y_0) + \left(\frac{b}{d}t, -\frac{a}{d}t\right) = (39, -29) + (82t, -61t) = (39 + 82t, -29 - 61t); \text{ for } t \in \mathbf{Z}$$

Hence, all solutions are of the form:

$$(x, y) = (39 + 82t, -29 - 61t); \text{ for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 39 + 82t; \quad y = -29 - 61t \quad \text{for } t \in \mathbf{Z}$$

4. (a) If $\gcd(a, b) = 1$, prove that $\gcd(a + b, a - b) = 1$ or 2 .

Proof. Suppose that $\gcd(a, b) = 1$.

Let $d = \gcd(a + b, a - b)$.

This means that d is a *common* divisor of $a + b$ and $a - b$, and hence, of their sum,
 $(a + b) + (a - b) = 2a$.

Similarly, d divides the difference of $a + b$ and $a - b$. (i.e., d divides $(a + b) - (a - b) = 2b$.)

Since d is a *common* divisor of $2a$ and $2b$, it follows that

$$d \leq \gcd(2a, 2b) = 2 \gcd(a, b) = 2 \cdot 1 = 2.$$

i.e., $d \leq 2$.

Hence, $d = \gcd(a + b, a - b) = 1$ or 2 . ■

4. (b) $\gcd(2a + b, a + 2b) = 1$ or 3 .

Proof.

Let $d = \gcd(2a + b, a + 2b)$.

This means that d is a *common* divisor of $2a + b$ and $a + 2b$, and hence, d divides any linear combination of $2a + b$ and $a + 2b$.

In particular, d divides $2(2a + b) - (a + 2b) = 3a$.

Also, d divides $-(2a + b) + 2(a + 2b) = 3b$.

i.e., d is a *common* divisor of $3a$ and $3b$.

$$\text{Hence, } d \leq \underbrace{\gcd(3a, 3b)}_{\text{By Thm 2.7}} = 3 \cdot \gcd(a, b) = 3 \cdot 1 = 3.$$

i.e., $d \leq 3$

So, at this point, we can say that $d = 1, 2$, or 3 .

But note that d *cannot* be equal to 2 .

To see this, note from above, that $\gcd(3a, 3b) = 3$.

Recall also, that d is a *common* divisor of $3a$ and $3b$.

Thus, by Thm 2.6, d must divide $\gcd(3a, 3b)$.

i.e., $d|3$.

Since $2 \nmid 3$, it follows that $d \neq 2$.

Hence, $d = 1$ or 3 . ■