MTH 3318 - Test #3

Fall 2022

Pat Rossi Name ________________________________

Instructions. Show your work completely. Document your work well.

Remark 1 *For problems 1 - 3, prove two.*

1. Prove or disprove: $f : \Re \longrightarrow \Re$ given by $f(x) = 3x^2 - 2$, is one to one.

   This is FALSE.

   As a counter-example, consider: $x_1 = -1$ and $x_2 = 1$

   Observe: $f(x_1) = 3x_1^2 - 2 = 3(-1)^2 - 2 = 1$ and $f(x_2) = 3x_2^2 - 2 = 3(1)^2 - 2 = 1$

   Note: $x_1 \neq x_2$, and yet, $f(x_1) = f(x_2)$

   Therefore, f is NOT one to one. ■

2. Prove or disprove: $f : \Re \longrightarrow \Re$ given by $f(x) = 5x + 1$, is one to one.

   Proof. Suppose that
$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow \quad 5x_1 + 1 &= 5x_2 + 1 \\ \Rightarrow \quad 5x_1 &= 5x_2 \\ \Rightarrow \quad x_1 &= x_2 \end{aligned}$$

   i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

   Hence, f is one to one. ■

3. Prove or disprove: $f : \Re \longrightarrow \Re$ given by $f(x) = x^3 + 3$, is one to one.

Proof. Suppose that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow \quad x_1^3 + 3 &= x_2^3 + 3 \\ \Rightarrow \quad x_1^3 &= x_2^3 \\ \Rightarrow \quad x_1 &= x_2 \end{aligned}$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, f is one to one. ■

Remark 2 *For problems 4 - 6, prove two.*

4. Prove or disprove: $f : \Re \longrightarrow \Re$ given by $f(x) = 3x^2 - 2$, is onto.

   This is FALSE.

   As a counter-example, consider: $y \in \mathbb{R}$ given by $y = -3$.

   Note that since $x^2 \geq 0$, it follows that $3x^2 - 2 \geq -2$.

   Therefore, given $y = -3$, there does not exist an x such that $f(x) = -3$

   Consequently, f is NOT onto. ■

5. Prove or disprove: $f : \Re \longrightarrow \Re$ given by $f(x) = 5x + 1$, is onto.

   Proof. We must show that $\forall y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$.

   Let $y \in \mathbf{R}$ be given.

   Let $x \in \mathbf{R}$ be given by $x = \underline{\frac{y-1}{5}}$

   Observe: $f(x) = \underline{5x+1} = \underline{5\left(\frac{y-1}{5}\right) + 1} = \underline{(y-1)+1} = y$.

   Thus, given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\frac{y-1}{5}}$) such that $f(x) = y$.

   Hence, $f(x)$ is onto. ■

Scratchwork

We want: x such that $f(x) = y$.
$\Rightarrow 5x + 1 = y$
$\Rightarrow 5x = y - 1$
$\Rightarrow x = \frac{y-1}{5}$

6. Prove or disprove: $f : \Re \longrightarrow \Re$ given by $f(x) = x^3 + 3$, is onto.

Proof. We must show that $\forall y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$.

Let $y \in \mathbf{R}$ be given.

Let $x \in \mathbf{R}$ be given by $x = \underline{\sqrt[3]{y-3}}$

Observe: $f(x) = \underline{x^3 + 3} = \underline{\left(\sqrt[3]{y-3}\right)^3 + 3} = \underline{(y-3)+3} = y$.

Thus, given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\sqrt[3]{y-3}}$) such that $f(x) = y$.

Hence, $f(x)$ is onto. ■

Scratchwork

We want: x such that $f(x) = y$.
$\Rightarrow x^3 + 3 = y$
$\Rightarrow x^3 = y - 3$
$\Rightarrow x = \sqrt[3]{y-3}$

Remark 3 *For problems 7-8.*

7. **Prove:** The set of odd natural numbers, $\mathbf{O} = \{1, 3, 5, 7, \ldots, 2n-1, \ldots\}$, is countably infinite (i.e., denumerable).

   Proof. Observe:
$$\begin{array}{cccccccccc} & \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots & \} \\ f & \downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ & \mathbf{O} & = & \{ & 1, & 3, & 5, & 7, & 9, & 11, & \ldots & \} \end{array}$$

   Define $f : \mathbf{N} \to \mathbf{O}$ by $f(n) = 2n - 1$.

   Clearly from the diagram above, f is one to one and onto. Hence, $\mathbf{O}$ is denumerable. ■

8. **Prove:** The set of Integers, $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$, is countably infinite (i.e., denumerable).

   Proof. Observe:
$$\begin{array}{ccccccccccccc} & \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots & \} \\ f & \downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ & \mathbf{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \ldots & \} \end{array}$$

   Define $f : \mathbf{N} \to \mathbf{Z}$ by $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

   Clearly from the diagram above, f is one to one and onto. Hence, $\mathbf{Z}$ is denumerable. ■

Remark 4 *For problems 9 - 10, prove either one.*

9. The set of positive rational numbers $\mathbb{Q}^+$ is countably infinite (i.e., denumerable).

   Consider the table of ordered pairs below:

$$\begin{array}{ccccccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) & & (1,5) & \rightarrow & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
(2,1) & & (2,2) & & (2,3) & & (2,4) & & (2,5) & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
(3,1) & & (3,2) & & (3,3) & & (3,4) & & (3,5) & \cdots & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
(4,1) & & (4,2) & & (4,3) & & (4,4) & & (4,5) & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
(5,1) & & (5,2) & & (5,3) & & (5,4) & & (5,5) & \cdots & \\
\vdots & \swarrow & \vdots & \nearrow & \vdots & \swarrow & \vdots & \nearrow & \vdots & &
\end{array}$$

   If we designate the ordered pair (i,j) in the ith row and jth column to represent the quotient of integers $\frac{i}{j}$, then every positive rational number appears in the table at least once. (e.g., the rational number $\frac{m}{n}$ appears in the mth row and n^{th} column.)

   Furthermore, the arrows in the table induce an exhaustive **ordering** of the positive rational numbers as follows:

$$\begin{array}{cccccccccccc}
1, & \frac{1}{2}, & 2, & 3, & \frac{1}{3}, & \frac{1}{4}, & \frac{2}{3}, & \frac{3}{2}, & 4, & 5, & \frac{1}{5}, & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
1^{st} & 2^{nd} & 3^{rd} & 4^{th} & 5^{th} & 6^{th} & 7^{th} & 8^{th} & 9^{th} & 10^{th} & 11^{th} &
\end{array}$$

   (Note that we have discarded repetitions of rationals if they occur. e.g., we have discarded $(2,2)$ because it is equivalent to $(1,1)$ which is already on our list.)

   But exhibiting an exhaustive ordering of the positive rationals is exactly the same as exhibiting a one to one correspondence between the natural numbers and the positive rational numbers.

$$\begin{array}{cccccccccccccccc}
 & \mathbb{Q}^+ & = & \{ & 1, & \frac{1}{2}, & 2, & 3, & \frac{1}{3}, & \frac{1}{4}, & \frac{2}{3}, & \frac{3}{2}, & 4, & 5, & \frac{1}{5}, & \cdots \ \} \\
f & \uparrow & & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
 & \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & \cdots \ \}
\end{array}$$

   Hence, the positive rational numbers are denumerable. ■

10. The set of real numbers in the interval $[0,1]$ is uncountable (i.e., non-denumerable).

Proof. (By contradiction)

Suppose, for the sake of deriving a contradiction, that the set of real numbers in the interval $[0,1]$ is denumerable.

Then there exists an *exhaustive ordering* of the set of real numbers in the interval $[0,1]$.

$$\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$$

Note that this ordering contains ALL of the real numbers in the interval $[0,1]$.

Consider the decimal expansions of these numbers:

$x_1 = 0.x_{11}x_{12}x_{13}\ldots$

$x_2 = 0.x_{21}x_{22}x_{23}\ldots$

$x_3 = 0.x_{31}x_{32}x_{33}\ldots$

$\vdots$

$x_n = 0.x_{n1}x_{n2}x_{n3}\ldots x_{nn}\ldots$

$\vdots$

Observe: Here, x_{ij} is the jth digit past the decimal point in the decimal expansion of the ith real number x_i.

Also: If x_i can be written in terminating and non-terminating form (e.g., 0.5 can be written as $0.499999\ldots$), then we choose the non-terminating form.

(The number 0 will be represented as $0.000\ldots$)

Define $y \in [0,1]$ as follows:

$y = 0.y_1y_2y_3\ldots y_{n\ldots}$ where y_i is the ith digit past the decimal point in the decimal expansion of y.

For $n = 1, 2, 3, \ldots$ define the digit y_n as follows:

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5 \\ 2 & \text{if } x_{nn} = 5 \end{cases}$$

Observe: $y \in [0,1]$ and yet $y \neq x_n$ for any $n \in \mathbf{N}$.

The reason for this is that, by construction of y, the n^{th} digit of y is different from the

n^{th} digit of x_n (i.e., $y_n \neq x_{nn}$) for all $n \in \mathbf{N}$.

Hence, $y \neq x_n \; \forall n \in \mathbf{N}$.

This contradicts our assumption that our list contains ALL of the real numbers in the interval $[0, 1]$.

Since the assumption that the set of real numbers in the interval $[0, 1]$ is denumerable led to this contradiction, the assumption must be false. Hence, the numbers in the interval $[0, 1]$ is non-denumerable (uncountable). ■

Remark 5 ***Select TWO problems from problems 11 - 15.***

11. Prove or disprove: $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow x + y \in \mathbf{Q}$

    Proof. Let $x, y \in \mathbf{Q}$.

    Then $\exists\, m, n, r, s \in \mathbf{Z}$ with $n, s \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{r}{s}$.

    Observe: $x + y = \frac{m}{n} + \frac{r}{s} = \frac{ms+nr}{ns}$.

    Since integers are closed under addition and multiplication, $x + y = \frac{ms+nr}{ns}$ is the quotient of integers, hence rational. ■

12. Prove or disprove: $x \in \mathbf{Q}$ and $y \in \mathbf{Q}^c \Rightarrow x + y \in \mathbf{Q}^c$

    Proof. Let $x \in \mathbf{Q}$, and $y \in \mathbf{Q}^c$.

    Suppose, for the sake of deriving a contradiction, that $x + y = z$, where $z \in \mathbf{Q}$.

    Then $y = \underbrace{z}_{\in \mathbf{Q}} - \underbrace{x}_{\in \mathbf{Q}} \Rightarrow y$ is rational, since it is the difference of rationals.

    This contradicts the fact that $y \in \mathbf{Q}^c$.

    Since the assumption that $z \in \mathbf{Q}$ leads to a contradiction, it must be the case that $z \in \mathbf{Q}^c$.

    Hence, $x + y \in \mathbf{Q}^c$. ■

13. Prove or disprove: $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow xy \in \mathbf{Q}$

    Proof. Let $x, y \in \mathbf{Q}$.

    Then $\exists\, m, n, r, s \in \mathbf{Z}$ with $n, s \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{r}{s}$.

    Observe: $x \cdot y = \frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns}$.

    Since integers are closed under multiplication, $x \cdot y = \frac{mr}{ns}$ is the quotient of integers, hence rational. ■

14. Prove or disprove: $x \in \mathbf{Q}$ and $y \in \mathbf{Q}^c \Rightarrow x + y \in \mathbf{Q}^c$

Proof. Let $x \in \mathbf{Q}$, and $y \in \mathbf{Q}^c$.

Suppose, for the sake of deriving a contradiction, that $x + y = z$, where $z \in \mathbf{Q}$.

Then $y = \underbrace{z}_{\in \mathbf{Q}} - \underbrace{x}_{\in \mathbf{Q}} \Rightarrow y$ is rational, since it is the difference of rationals.

This contradicts the fact that $y \in \mathbf{Q}^c$.

Since the assumption that $z \in \mathbf{Q}$ leads to a contradiction, it must be the case that $z \in \mathbf{Q}^c$.

Hence, $x + y \in \mathbf{Q}^c$. ■

15. Prove or disprove: $x \in \mathbf{Q}^c$ and $y \in \mathbf{Q}^c \Rightarrow xy \in \mathbf{Q}^c$

Proof. This is FALSE.

As a counter-example, consider $x = \sqrt{2}$ and $y = \sqrt{2}$

Observe: $x \in \mathbf{Q}^c$ and $y \in \mathbf{Q}^c$

However, $xy = \sqrt{2} \cdot \sqrt{2} = 2 \notin \mathbf{Q}^c$

Consequently, $x \in \mathbf{Q}^c$ and $y \in \mathbf{Q}^c \not\Rightarrow xy \in \mathbf{Q}^c$ ■